Regular Nilpotent Hessenberg Varieties: Fixed Points and Equivariant Cohomology

Elizabeth Drellich

University of Massachusetts, Amherst

1 Hessenberg Varieties

A **Hessenberg variety** is a subvariety of the flag variety determined by

- X, an $n \times n$ matrix, and
- $\mathcal{H}$, a subspace of $n \times n$ matrices containing the upper triangular matrices that has a staircase boundary. Equivalently $h : [n] \to [n]$ an increasing function with $h(i) \geq i$ for all i, with matrix basis unit $E_{(i,j)} \in \mathcal{H}$ whenever $i \leq h(j)$

The Hessenberg variety $\mathcal{X}(\mathcal{H}, X)$ is

$$\begin{aligned}\mathcal{X}(\mathcal{H}, X) &= \{gB \in Gl_n(\mathbb{C})/B : g^{-1}Xg \in \mathcal{H}\} \\ &= \{\mathcal{F}_\bullet = V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{C}^n : XV_i \subseteq V_{h(i)} \text{ for all } i\}\end{aligned}$$

A **regular nilpotent Hessenberg variety** is $\mathcal{X}_\mathcal{H} = \mathcal{X}(\mathcal{H}, N)$ where N is the nilpotent matrix with one Jordan block. The variety $\mathcal{X}_\mathcal{H}$ is in general irreducible and not smooth. There is an S^1 action on $\mathcal{X}_\mathcal{H}$ with fixed points $gB \in \mathcal{X}_\mathcal{H}$ where g is a permutation matrix. Each fixed point can be written as a product of simple reflections $\{s_i = (i, i+1)\} \in S_n$ and the set of fixed points is denoted

$$W_\mathcal{H} = \{v \in S_n : v^{-1}Xv \in \mathcal{H}\}.$$

Example.

$$N = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}, \quad \begin{matrix} h(1) = 2 \\ h(2) = 2 \\ h(3) = 4 \\ h(4) = 5 \\ h(5) = 5 \end{matrix}, \quad \mathcal{H} = \begin{bmatrix} *&*&*&*&* \\ *&*&*&*&* \\ 0&0&*&*&* \\ 0&0&*&*&* \\ 0&0&0&*&* \end{bmatrix}, \quad W_\mathcal{H} = \left\{ \begin{matrix} 1, s_3, s_3s_4s_3, s_4, \\ s_1, s_1s_3, s_1s_3s_4s_3, s_1s_4 \end{matrix} \right\}$$

2 The Relationship between $\mathcal{H}$ and $\mathcal{H}^\perp$

How does $\mathcal{X}_\mathcal{H}$ compare to $\mathcal{X}_{\mathcal{H}^\perp}$ where $\mathcal{H}^\perp$ is $\mathcal{H}$ flipped along its antidiagonal?

Example.

$$\mathcal{H} = \begin{bmatrix} *&*&*&*&* \\ *&*&*&*&* \\ 0&0&*&*&* \\ 0&0&*&*&* \\ 0&0&0&*&* \end{bmatrix}, \quad \mathcal{H}^\perp = \begin{bmatrix} *&*&*&*&* \\ *&*&*&*&* \\ 0&*&*&*&* \\ 0&0&0&*&* \\ 0&0&0&*&* \end{bmatrix}$$

Theorem (D). *For any shape $\mathcal{H}$*

$$\mathcal{X}_\mathcal{H} \cong \mathcal{X}_{\mathcal{H}^\perp}$$

and the fixed points are related by

$$W_{\mathcal{H}^\perp} = w_0 W_\mathcal{H} w_0$$

where $w_0 \in S_n$ is the permutation $w_0(i) = n - i + 1$ for all $i \in [n]$.

Proof. (Sketch) The map

$$\begin{aligned} GL_n(\mathbb{C})/B &\to GL_n(\mathbb{C})/B \\ gB &\mapsto w_0(g^\top)^{-1}w_0B \end{aligned}$$

restricts to a homeomorphism between $\mathcal{X}_\mathcal{H}$ and $\mathcal{X}_{\mathcal{H}^\perp}$. □

3 Schubert Classes and Billey's Formula

Schubert varieties are a class of subvarieties of the flag variety whose fundamental classes induce a well-known basis of the cohomology of the flag variety. There is a standard torus action of T on the flag variety with permutation flags as fixed points. This action restricts to Schubert varieties. The combinatorics of the permutations encodes geometric data about the variety including dimension, smoothness, and singular loci.

The set of **equivariant Schubert classes** $\{\sigma_v\}$ forms a basis of the equivariant cohomology of the flag variety. Each class in $H^*_T(GL_n(\mathbb{C})/B)$ can be thought of as a collection of $n!$ polynomials from $\mathbb{C}[t_1, \ldots, t_n]$ corresponding to elements in S_n via the localization map

$$H^*_T(GL_n(\mathbb{C})/B) \to H^*_T(\{T\text{-fixed points of } GL_n(\mathbb{C})/B\})$$

Billey gave an explicit formula for the localizations $\sigma_v(w)$ of the equivariant Schubert class σ_v at the permutation flag wB.

Billey's Formula*.* Fix a reduced word for $w = s_{b_1}s_{b_2}\cdots s_{b_{\ell(w)}}$ and let

$$\mathbf{r}(\mathbf{i}, w) = s_{b_1}s_{b_2}\cdots s_{b_{i-1}}(t_{b_i} - t_{b_i+1}).$$

These terms are polynomials of degree 1 and Billey's formula is

$$\sigma_v(w) = \sum_{\substack{\text{reduced words} \\ v = s_{b_{j_1}}s_{b_{j_2}}\cdots s_{b_{j_{\ell(v)}}}}} \left(\prod_{i=1}^{\ell(v)} \mathbf{r}(\mathbf{j_i}, w) \right).$$

This is a polynomial of degree $\ell(v)$ in n variables with non-negative integer coefficients.

4 Harada and Tymoczko's Conjecture

Harada and Tymoczko conjecture that a similar construction will form a basis for the equivariant cohomology $H^*_{S^1}(\mathcal{X}_\mathcal{H})$. They conjecture that for the set $V_\mathcal{H} := S_n \cap \mathcal{H}$, the equivariant Schubert classes $\tilde{\sigma}_v$ for $v \in V_\mathcal{H}$ localized at the points $w \in W_\mathcal{H}$ form a basis for the equivariant cohomology $H^*_{S^1}(\mathcal{X}_\mathcal{H})$. The class $\tilde{\sigma}_v$ has localizations $\tilde{\sigma}_v(w) = \phi(\sigma_v(w))$ where $\phi(t_i) = -it$.

To show that the conjecture is true for a shape $\mathcal{H}$ it is sufficient to show that the determinant of $\tilde{A}_\mathcal{H}$ is non zero, where

$$\tilde{A}_\mathcal{H} = (\tilde{\sigma}_v(w))_{\substack{w \in W_\mathcal{H} \\ v \in V_\mathcal{H}}}$$

We verified this by computer for all shapes $\mathcal{H}$ for $n \leq 5$. It has been proven for certain shapes $\mathcal{H}$ for all n including

Reg. Nil. Springer Variety

$$\begin{bmatrix} *&*&*&*&*&* \\ 0&*&*&*&*&* \\ 0&0&*&*&*&* \\ 0&0&0&*&*&* \\ 0&0&0&0&*&* \\ 0&0&0&0&0&* \end{bmatrix}$$

Peterson Variety

$$\begin{bmatrix} *&*&*&*&*&* \\ *&*&*&*&*&* \\ 0&*&*&*&*&* \\ 0&0&*&*&*&* \\ 0&0&0&*&*&* \\ 0&0&0&0&*&* \end{bmatrix}$$

Harada-Tymoczko

Modified Peterson Variety

$$\begin{bmatrix} *&*&*&*&*&* \\ *&*&*&*&*&* \\ *&*&*&*&*&* \\ 0&0&*&*&*&* \\ 0&0&0&*&*&* \\ 0&0&0&0&*&* \end{bmatrix}$$

Bayegan-Harada

Proving this conjecture will put regular nilpotent Hessenberg varieites into the broader context of Schubert calculus by identifying a combinatorially natural basis of Schubert classes for the equivariant cohomology $H^*_{S^1}(\mathcal{X}_\mathcal{H})$.

5 Decomposable Hessenberg Varieties

A regular nilpotent Hessenberg variety $\mathcal{X}_\mathcal{H}$ is **decomposable** if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Example.

$$\mathcal{H} = \begin{bmatrix} *&*&*&*&* \\ *&*&*&*&* \\ 0&0&*&*&* \\ 0&0&*&*&* \\ 0&0&0&*&* \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1 & \begin{matrix} *&*&* \\ *&*&* \end{matrix} \\ \begin{matrix} 0&0 \\ 0&0 \\ 0&0 \end{matrix} & \mathcal{H}_2 \end{bmatrix}$$

Theorem (D). *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $\tilde{A}_\mathcal{H} = \tilde{A}_{\mathcal{H}_1} \otimes \tilde{A}_{\mathcal{H}_2}$. Therefore $\tilde{A}_\mathcal{H}$ has a non zero determinant if and only if both $\tilde{A}_{\mathcal{H}_1}$ and $\tilde{A}_{\mathcal{H}_2}$ do.*

Proof. (Sketch)

- If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $W_\mathcal{H} = W_{\mathcal{H}_1} \oplus W_{\mathcal{H}_2}$ and $V_\mathcal{H} = V_{\mathcal{H}_1} \oplus V_{\mathcal{H}_2}$.
- If $w = w_1 \oplus w_2$ for $w_1 \in W_{\mathcal{H}_1}$ and $w_2 \in W_{\mathcal{H}_2}$ and $v = v_1 \oplus v_2$ for $v_1 \in V_{\mathcal{H}_1}$ and $v_2 \in V_{\mathcal{H}_2}$ then Billey's formula has the property that $\sigma_v(w) = \sigma_{v_1}(w_1)\sigma_{v_2}(w_2)$.
- Together these show that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $\tilde{A}_\mathcal{H} = \tilde{A}_{\mathcal{H}_1} \otimes \tilde{A}_{\mathcal{H}_2}$.

□

Example. Using the shape $\mathcal{H}$ above

$$W_\mathcal{H} = \left\{ \begin{matrix} 1, s_3, s_3s_4s_3, s_4 \\ s_1, s_1s_3, s_1s_3s_4s_3, s_1s_4 \end{matrix} \right\} \qquad V_\mathcal{H} = \left\{ \begin{matrix} 1, s_3, s_4s_3, s_4 \\ s_1, s_1s_3, s_1s_4s_3, s_1s_4 \end{matrix} \right\}$$

$$W_{\mathcal{H}_1} = \{1, s_1\} \qquad V_{\mathcal{H}_1} = \{1, s_1\}$$

$$W_{\mathcal{H}_2} = \{1, s_3, s_3s_4s_3, s_4\} \qquad V_{\mathcal{H}_2} = \{1, s_3, s_3s_4, s_4\}$$

$$\sigma_{s_1s_3s_4}(s_1s_3s_4s_3) = (t_1 - t_2)(t_3 - t_4)(t_3 - t_5) = \sigma_{s_1}(s_1)\sigma_{s_3s_4}(s_3s_4s_3)$$

$$\tilde{A}_\mathcal{H} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 1&t&0&0&0&0&0&0 \\ 1&2t&2t^2&2t&0&0&0&0 \\ 1&0&0&t&0&0&0&0 \\ 1&0&0&0&t&0&0&0 \\ 1&t&0&0&t&t^2&0&0 \\ 1&2t&2t^2&2t&t&2t^2&2t^3&2t^2 \\ 1&0&0&t&t&0&0&t^2 \end{bmatrix} = \overset{\tilde{A}_{\mathcal{H}_1}}{\begin{bmatrix} 1&0 \\ 1&t \end{bmatrix}} \otimes \overset{\tilde{A}_{\mathcal{H}_2}}{\begin{bmatrix} 1&0&0&0 \\ 1&t&0&0 \\ 1&2t&2t^2&2t \\ 1&0&0&t \end{bmatrix}}$$

6 References

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