A nonabelian Hodge theorem for twisted vector bundles

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The nonabelian Hodge theorem

Let X be a smooth projective variety over $\mathbb{C}$. The nonabelian Hodge theorem of [Sim92] gives a fully faithful functor of groupoids

$$\left\{\begin{array}{c}\text{Flat vector bundles}\\ \text{of rank } n \text{ on } X\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{Higgs bundles}\\ \text{of rank } n \text{ on } X\end{array}\right\}$$

Here a *flat vector bundle* on X is a pair $(\mathcal{E}, \nabla)$ of a vector bundle $\mathcal{E}$ together with a flat connection ∇ on it. A *Higgs bundle* $(\mathcal{F}, \phi)$ is a vector bundle $\mathcal{F}$ equipped with an $\mathcal{O}_X$-linear map $\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$ with curvature $C(\phi) = -\phi \wedge \phi = 0$.

Twisted vector bundles

Given a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X and a Čech 2-cocycle $\underline{\alpha} = \{\alpha_{ijk}\} \in \check{Z}^2(\mathfrak{U}, \mathcal{O}_X^\times)$, an $\underline{\alpha}$*-twisted vector bundle* [Gir71, Căl00] is a collection $\big(\{\mathcal{E}_i\}, \{g_{ij}\}\big)$ of vector bundles $\mathcal{E}_i$ on U_i and isomorphisms $g_{ij} : \mathcal{E}_j\big|_{U_{ij}} \to \mathcal{E}_i\big|_{U_{ij}}$ satisfying the twisted cocycle condition $g_{ij}g_{jk}g_{ki} = \alpha_{ijk}$. Morphisms of $\underline{\alpha}$-twisted vector bundles are isomorphisms of the locally defined vector bundles that intertwine the transition functions.

Adding connections

Choose cochains $\{\omega_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X)$ and $\{F_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X)$, and equip each $\mathcal{E}_i$ in an $\underline{\alpha}$-twisted vector bundle with a connection ∇_i satisfying

$$\nabla_i - g_{ij}\nabla_j g_{ij}^{-1} = \omega_{ij}, \qquad C(\nabla_i) = F_i$$

The compatibility conditions make the triple $\big(\{\alpha_{ijk}\}, \{\omega_{ij}\}, \{F_i\}\big)$ into a 2-cocycle in hypercohomology of the multiplicative de Rham complex

$$\mathrm{dR}_X^\times := \Big[\mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots\Big]$$

We call the latter triple a $\mathfrak{U}$-$\mathbb{G}_m$*-gerbe with flat connection*, and say that $\big(\{\mathcal{E}_i\}, \{\nabla_i\}, \{g_{ij}\}\big)$ is a vector bundle on it.

Adding Higgs fields

Choose cochains $\{\omega'_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X)$ and $\{F'_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X)$, and endow each $\mathcal{E}'_i$ in an $\underline{\alpha}'$-twisted vector bundle with a Higgs field ϕ_i such that

$$\phi_i - g'_{ij}\phi_j(g'_{ij})^{-1} = \omega'_{ij}, \qquad C(\phi_i) = F'_i$$

The triple $\big(\{\alpha'_{ijk}\}, \{\omega'_{ij}\}, \{F'_i\}\big)$ assembles into a 2-cocycle in hypercohomology of the multiplicative Dolbeault complex

$$\mathrm{Dol}_X^\times := \Big[\mathcal{O}_X^\times \xrightarrow{0} \Omega^1_X \xrightarrow{0} \Omega^2_X \xrightarrow{0} \cdots\Big]$$

We call this a *Higgs* $\mathfrak{U}$-$\mathbb{G}_m$*-gerbe*, and the triple $\big(\{\mathcal{E}'_i\}, \{\phi_i\}, \{g'_{ij}\}\big)$ a vector bundle on it.

Main theorem (the cocycle picture)

Given a $\mathfrak{U}$-$\mathbb{G}_m$-gerbe with flat connection $\big(\{\alpha_{ijk}\}, \{\omega_{ij}\}, \{F_i\}\big)$, there is a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\big(\{\alpha'_{ijk}\}, \{\omega'_{ij}\}, \{F'_i\}\big)$ such that we have a fully faithful functor

$$\left\{\begin{array}{c}\text{Vector bundles of rank } n\\ \text{on } \big(\{\alpha_{ijk}\}, \{\omega_{ij}\}, \{F_i\}\big)\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{Vector bundles of rank } n\\ \text{on } \big(\{\alpha'_{ijk}\}, \{\omega'_{ij}\}, \{F'_i\}\big)\end{array}\right\}$$

Conversely, given a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\big(\{\alpha'_{ijk}\}, \{\omega'_{ij}\}, \{F'_i\}\big)$, there is a $\mathfrak{U}$-$\mathbb{G}_m$-gerbe with flat connection $\big(\{\alpha_{ijk}\}, \{\omega_{ij}\}, \{F_i\}\big)$ such that the same conclusion holds.

Codifying twisted vector bundles: gerbes

$\mathbb{G}_m$-gerbes [Gir71] over X (with band in the terminology of *loc.cit.*) are certain 1-stacks locally isomorphic to $B\mathbb{G}_m$. They provide geometric representatives of classes in $H^2(X, \mathcal{O}_X^\times)$. As principal $B\mathbb{G}_m$-bundles [NSS12], we have a classifying (2-)stack for them:

$$\begin{array}{ccc} {}_\alpha X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\alpha} & B^2\mathbb{G}_m \end{array}$$

We can realize $\underline{\alpha}$-twisted vector bundles on X as those vector bundles on the gerbe ${}_\alpha X$ classified by $\alpha = [\underline{\alpha}] \in H^2(X, \mathcal{O}_X^\times)$ whose classifying morphism ${}_\alpha X \to BGL_n$ is $B\mathbb{G}_m$-equivariant. We have:

$${}_{B\mathbb{G}_m}\big({}_\alpha X, BGL_n\big) \simeq \big(X, B\mathbb{P}GL_n\big) \times^h_{(X, B^2\mathbb{G}_m), \alpha} *$$

Notation: for X and Y objects in an ∞-topos, (X, Y) denotes the ∞-groupoid of morphisms from X to Y. If they come equipped with an action of G, we denote by ${}_G(X, Y)$ the ∞-groupoid of G-equivariant morphisms.

Codifying connections: the de Rham stack

The *de Rham stack* X_{dR} of X is the quotient of X by the formal neighborhood of the diagonal in $X \times X$. It encodes flat connections:

- vector bundles on it are flat vector bundles on X;
- we call $\mathbb{G}_m$-gerbes over it $\mathbb{G}_m$*-gerbes with flat connection* over X.

Equivalence classes of the latter are given by $\mathbb{H}^2(X, \mathrm{dR}_X^\times)$.

Given a $\mathbb{G}_m$-gerbe over X_{dR}, vector bundles over it (with an equivariance condition) can be interpreted as twisted vector bundles with connection compatible with that of the gerbe:

$${}_{B\mathbb{G}_m}\big({}_\theta(X_{\mathrm{dR}}), BGL_n\big) \simeq \big(X_{\mathrm{dR}}, B\mathbb{P}GL_n\big) \times^h_{(X_{\mathrm{dR}}, B^2\mathbb{G}_m), \theta} *$$

Codifying Higgs fields: the Dolbeault stack

The *Dolbeault stack* X_{Dol} of X is the quotient of X by the formal neighborhood of the zero section of the tangent bundle of X. It encodes Higgs fields:

- vector bundles on it are Higgs bundles on X;
- we call $\mathbb{G}_m$-gerbes over it *Higgs* $\mathbb{G}_m$*-gerbes* over X.

Equivalence classes of the latter are given by $\mathbb{H}^2(X, \mathrm{Dol}_X^\times)$.

Given a $\mathbb{G}_m$-gerbe over X_{Dol}, vector bundles over it (with an equivariance condition) can be interpreted as twisted vector bundles with a Higgs field compatible with the Higgs data of the gerbe:

$${}_{B\mathbb{G}_m}\big({}_{\theta'}(X_{\mathrm{Dol}}), BGL_n\big) \simeq \big(X_{\mathrm{Dol}}, B\mathbb{P}GL_n\big) \times^h_{(X_{\mathrm{Dol}}, B^2\mathbb{G}_m), \theta'} *$$

Main theorem (intrinsic formulation)

Given $\theta \in (X_{\mathrm{dR}}, B^2\mathbb{G}_m)$ there exists $\theta' \in (X_{\mathrm{Dol}}, B^2\mathbb{G}_m)$ such that we have a fully faithful functor

$${}_{B\mathbb{G}_m}\big({}_\theta(X_{\mathrm{dR}}), BGL_n\big) \hookrightarrow {}_{B\mathbb{G}_m}\big({}_{\theta'}(X_{\mathrm{Dol}}), BGL_n\big)$$

Conversely, given $\theta' \in (X_{\mathrm{Dol}}, B^2\mathbb{G}_m)$ there is $\theta \in (X_{\mathrm{dR}}, B^2\mathbb{G}_m)$ such that the same conclusion holds.

The proof

The projective bundles on both sides are related by the nonabelian Hodge theorem [Sim92]. The gerbes are n-torsion (otherwise the two sides are empty and the correspondence is trivial), and results of [GR11] allow us to relate them using the higher nonabelian Hodge correspondence of [Sim02].

References

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