On $(2,4)$ Calabi-Yau Complete Intersections that contain an Enriques Surface

Lev Borisov and Howard Nuer
Department of Mathematics, Rutgers University, Piscataway, New Jersey

Introduction

Since the discovery of Superstring Theory and Mirror Symmetry, the construction and study of families of Calabi-Yau manifolds, and especially CY threefolds, has been an active and important branch of algebraic geometry. In their paper [GP], Gross and Popescu proved that a Calabi-Yau 3-fold that contains an Abelian surface admits a fibration over $\mathbb{P}^1$ by such threefolds, and they constructed a number of families of such 3-folds. In [BN] we prove that a CY 3-fold $\mathbf{X}$ containing an Enriques $\mathbf{S}$ surface admits a similar fibration over $\mathbb{P}^1$, this time by K3 surfaces, with a double fiber whose reduction is $\mathbf{S}$. We also construct 4 different such families, and for those coming from $(2,4)$ complete intersections, we classify all of its smooth birational models.

Fibration Lemma

If a Calabi-Yau 3-fold $\mathbf{X}$ contains an Enriques surface $\mathbf{S}$, then it admits a fibration $\pi : \mathbf{X} \longrightarrow \mathbb{P}^1$ induced by the linear system $|2\mathbf{S}|$ with a double fibre along $\mathbf{S}$ and generic fiber a K3 surface.

Construction

Consider a generic Reye Enriques surface $\mathbf{S}$ in its Reye Fano embedding into $\mathbf{G(2,4)} \subset \mathbb{P}^5$. Intersect $\mathbf{G(2,4)}$ with a generic cubic fourfold containing $\mathbf{S}$, and let $\mathbf{X}$ be the resulting $(2,4)$ complete intersection 3-fold. Then:

- $\mathbf{X}$ is an irreducible 3-fold whose singular locus consists of 58 nodes, all of which sit along $\mathbf{S}$;
- $\mathbf{X}$ has a small resolution $\pi_1 : \mathbf{X}^1 \longrightarrow \mathbf{X}$ obtained by blowing-up $\mathbf{S}$;
- $\mathbf{X}$ has another small resolution $\pi_0 : \mathbf{X}^0 \longrightarrow \mathbf{X}$ obtained by flopping the 58 exceptional curves of π_1. $\mathbf{X}^0$ admits a K3 fibration over $\mathbb{P}^1$ with double fibre along $\mathbf{S}$;
- The K3 surfaces occuring in the fibration of $\mathbf{X}^0$ above are actually determinantal quartics in $\mathbb{P}^3$;
- Both $\mathbf{X}^1$ and $\mathbf{X}^0$ are smooth CY 3-folds such that $\chi(\mathbf{X}^1) = \chi(\mathbf{X}^0) = -60$, $\mathbf{h}^{1,1}(\mathbf{X}^1) = \mathbf{h}^{1,1}(\mathbf{X}^0) = 2$, and $\mathbf{h}^{2,1}(\mathbf{X}^1) = \mathbf{h}^{2,1}(\mathbf{X}^0) = 32$.

It is worth noting that this construction gives all $(2,4)$ complete intersections containing an Enriques in its Fano embedding, since such an Enriques must be Reye.

Alternate Description

A sometimes more convenient description of the nodal 3-fold $\mathbf{X}$ above is as follows: Let $\mathbf{Q}$ be the universal quotient bundle on $\mathbf{G(2,4)}$. Take a generic 4-dimensional subspace of the global sections of $\mathbf{Q} \otimes \mathbf{Q}$, with basis $\mathbf{s_1}, \dots, \mathbf{s_4}$. Then $\mathbf{X}$ is the determinantal hypersurface of $\mathbf{G(2,4)}$ given by $\det(\mathbf{s_1}, \dots, \mathbf{s_4}) = 0$.

A Nodal Birational Model in $\mathbb{P}(1,1,1,1,2,2)$

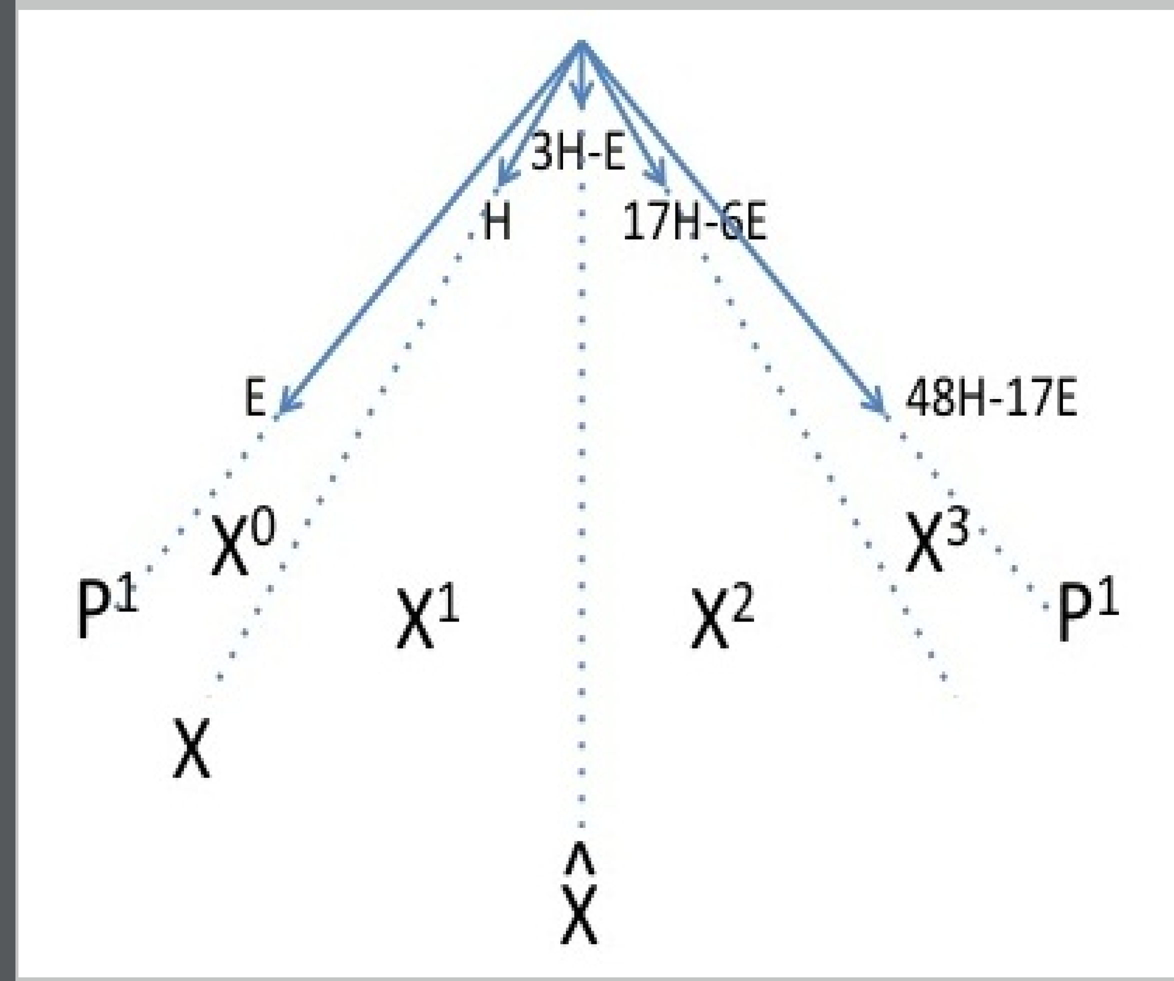
Let $\mathbf{A_i}$ (resp. $\mathbf{B_i}$) be the symmetric (resp. anti-symmetric) component of the section $\mathbf{s_i}$ above. Let $\mathbf{u_1}, \dots, \mathbf{u_4}, \mathbf{y}, \mathbf{z}$ be the homogeneous coordinates on $\mathbb{P}(1,1,1,1,2,2)$ with the respective wieghts, and consider the $(4,4)$ complete intersection $\hat{\mathbf{X}}$ given by the equations,

$$\begin{cases} \mathbf{y}^2 = \det(\sum_i \mathbf{u_i A_i}) \\ \mathbf{z}^2 = -\det(\sum_i \mathbf{u_i}(\mathbf{A_i} + \mathbf{B_i})) + (\mathbf{y} - \mathrm{Pf}(\sum_i \mathbf{u_i B_i}))^2 \end{cases} \quad (1)$$

of degree $\mathbf{4}$ in $\mathbb{P}(1,1,1,1,2,2)$. Then if $\mathbf{H} = \pi_1^* \mathcal{O}_{\mathbf{X}}(1)$, $\mathbf{E} = \pi_1^{-1}(\mathbf{S}) \in \mathbf{N}^1(\mathbf{X}^1)$, we find that the ray $\mathbb{R}_{\geq 0}(\mathbf{3H} - \mathbf{E}) \subset \mathbf{N}^1(\mathbf{X}^1)_{\mathbb{R}}$ induces a small resolution $\mathbf{X}^1 \rightarrow \hat{\mathbf{X}}$ of the precisely 42 nodal points of $\hat{\mathbf{X}}$ such that:

- We can flop the 42 exceptional $\mathbb{P}^1$'s to get another smooth birational model $\mathbf{X}^2$
- $\hat{\mathbf{X}}$ has the natural involution $\sigma : \mathbf{z} \mapsto -\mathbf{z}$, which fixes 22 of the nodes and lifts to a birational automorphism of $\mathbf{X}^1$ inducing the map $(\mathbf{H}, \mathbf{E}) \mapsto (\mathbf{17H} - \mathbf{6E}, \mathbf{48H} - \mathbf{17E})$ on $\mathbf{N}^1(\mathbf{X}^1)$.

Birational Models and the Movable Cone



We thus find that τ induces a nontrivial isomorphism between $\mathbf{X}^1$ and $\mathbf{X}^2$. So the ray generated by $\mathbf{17H} - \mathbf{6E}$ induces a contraction of 58 $\mathbb{P}^1$'s as before, and upon flopping these we find a fourth birational model $\mathbf{X}^3$ (necessarily isomorphic to $\mathbf{X}^0$ but nontrivially) which admits a K3 fibration with $\mathbf{48H} - \mathbf{17E}$ an Enriques surface which is the reduction of a double fibre. Thus the birational models of $\mathbf{X}$ look as in the figure. Moreover, the moveable cone, $\overline{\mathrm{Mov}}(\mathbf{X})$, is the convex cone generated by $\mathbf{E}$ and $\mathbf{48H} - \mathbf{17E}$. Here the $\mathbf{X^i}$ are depicted inside their nef cones.

Degenerations of our family and Mirror Symmetry

Consider the following very specific sections of $\mathbf{Q} \otimes \mathbf{Q}$:

$$\mathbf{s_1} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{s_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{s_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}, \quad \mathbf{s_4} = \begin{pmatrix} d & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & a \end{pmatrix},$$

with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}$ generic.

- The corresponding determinantal hypersurface of $\mathbf{G(2,4)}$ actually has 106 nodal points and crepant resolutions giving smooth Calabi-Yau 3-folds similar to those above, but with hodge numbers $(23,5)$ that are new.
- If we restrict further to the subfamily $\mathbf{a} = \mathbf{d}$, then we get a new family of CY 3-folds with hodge numbers $(31,1)$. This may provide the mirror family to a known family of CY 3-folds!.

$(3,3)$ CICY's and generic Enriques surfaces

If we consider the Fano embedding of any Enriques surface $\mathbf{S}$ into $\mathbb{P}^5$, then we may form a $(3,3)$ complete intersection $\mathbf{X}$ from two generic cubic fourfolds that contain $\mathbf{S}$, and we find crepant resolutions $\mathbf{X}^1$ and $\mathbf{X}^0$ in the same way as above, except that now we have 48 nodes, and the hodge numbers are $(2,26)$. For those $\mathbf{S}$ which sit inside the divisor of the moduli space of Enriques surfaces consisting of nodal Enriques surfaces (i.e. those with (-2)-curves), their movable cones are precisely half of the cone pictured above. That is, $\overline{\mathbf{Mov}}(\mathbf{X})$ is the convex cone generated by $\mathbf{E}$ and $\mathbf{3H} - \mathbf{E}$, as the ray $\mathbf{3H} - \mathbf{E}$ now induces a primitive contraction of type III, which is related to the beautiful geometry of Enriques surfaces. The movable cone for these CY's coming from generic Enriques Surfaces, however, may be larger than just this.

References

[BN] Borisov, L., Nuer, H., "On $(2,4)$ Complete Intersection Threefolds that Contain an Enriques Surfaces"
[GP] Gross, M., Popescu, S., "Calabi-Yau Threefolds and Moduli of Abelian Surfaces I"