# Geometry of Moduli Space of Minimal Dominant Rational Curves on Algebraic Varieties

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## Questions

- How to characterize a projective variety to be a complete intersection?
- Consider the general fiber $\mathcal{F}$ of the following evaluation map $ev$ of Kontsevich moduli space.

$$ev : \overline{\mathcal{M}}_{0,m}(X, m) \to X^m$$

What is the geometry of $\mathcal{F}$?

## Notations and Conventions

- Let $X$ be a smooth complete intersection of type $(d_1, d_2, \ldots, d_c)$ in $\mathbb{P}^n$
- The Kontsevich moduli stack $\overline{\mathcal{M}}_{0,m}(X, e)$ parameterizes data $(C, f, x_1, \ldots, x_m)$ of
  (i) a proper, connected, at-worst-nodal, arithmetic genus 0 curve $C$,
  (ii) an ordered collection $x_1, \ldots, x_m$ of distinct smooth points of $C$,
  (iii) and a morphism $f : C \to X$ whose image has degree $e$ in $\mathbb{P}^n$
  such that $(C, f, x_1, \ldots, x_m)$ has only finitely many automorphisms.
- There is an evaluation morphism

$$\text{ev} : \overline{\mathcal{M}}_{0,m}(X, e) \to X^m,$$
$$(C, f, x_1, \ldots, x_m) \mapsto (f(x_1), \ldots, f(x_m))$$

- Let $n, m, c, d_1, \ldots, d_c$ be numbers such that $n, m, c, d_i \in \mathbb{N}$, $n \geq m$, $c \leq n$, and $d_i \geq 2$.

## Main Theorem 1

Suppose $\Delta$ and $\mathcal{F}$ are smooth projective varieties. If the following conditions are satisfied,

- $\Delta \subseteq \mathcal{F} \subseteq \mathbb{P}^N$, $\Delta$ is a reduced divisor of a smooth projective variety $\mathcal{F}$.
- $\dim \Delta \geq 1$
- $\Delta$ is a complete intersection in $\mathbb{P}^N$ of type $(d_1, \ldots, d_c)$ where $d_i \geq 1$
- $\Delta$ is defined by a homogeneous polynomial of degree $d_1$ restricted to $\mathcal{F}$.

then $\mathcal{F}$ is a complete intersection of type $(d_2, \ldots, d_c)$ in $\mathbb{P}^N$.

## Main Theorem 2

Suppose $X$ is a smooth complete intersection variety ,but not a quadratic hypersurface, of type $(d_1, \ldots, d_c)$, $(d_i \geq 2)$ in $\mathbb{P}^n$. If we have $m = 2$ and $n \geq 2\sum d_i - c + 1$, then the general fiber $\mathcal{F}$ of the evaluation map ev is a smooth complete intersection variety in $\mathbb{P}^{n-2}$ of type

$$(1, 1, 2, 2, \ldots, d_1 - 1, d_1 - 1, d_1, \ldots, 1, 1, 2, 2, \ldots, d_c - 1, d_c - 1, d_c)$$
$$-(1, 1, 2).$$

## Main Theorem 3

Suppose X is not a quadratic hypersurface, $m \geq 3$, and $n + m(c - \sum d_i) - c \geq 1$. Let the forgetful functor be $F : \mathcal{F} \to \overline{\mathcal{M}}_{0,m}$. For a general point $t \in \overline{\mathcal{M}}_{0,m}$, we have a line bundle $\lambda|_{\mathcal{F}_t}$ on the fiber $\mathcal{F}_t$ of $F$ over $t$. Then, the complete linear system $|\lambda|_{\mathcal{F}_t}|$ of $\lambda_{\mathcal{F}_t}$ defines a map

$$|\lambda|_{\mathcal{F}_t}| : \mathcal{F}_t \hookrightarrow \mathbb{P}^N = \mathbb{P}^{n-m(c-1)}$$

Via this map, the smooth variety $\mathcal{F}_t$ is a complete intersection in $\mathbb{P}^N$ of type

$$T_1(d_1, m) = \begin{pmatrix} 2 \ldots d_1 - 1 & \\ \vdots \quad\quad \vdots & d_1 \\ 2 \ldots d_1 - 1 & \end{pmatrix}, \ldots, T_1(d_c, m) = \begin{pmatrix} 2 \ldots d_c - 1 & \\ \vdots \quad\quad \vdots & d_c \\ 2 \ldots d_c - 1 & \end{pmatrix}$$

## Reference


[1] *A. J. de Jong, J. Starr, Low Degree Complete Intersections Are Rationally Simply Connected*, preprint, 2006.

[2] *A. J. de Jong, J. Starr, Higher Fano manifolds and rational surfaces*, Duke Math. J. 139 (2007), no.1, 173-183. MR MR2322679 (2008j:14078)

[3] *C. Araujo, A. Castravet, Polarized Minimal Families of Rational Curves and Higher Fano Manifold* American Journal of Mathematics, Volume 134, Number 1, February 2012 pp. 87-107 | 10.1353/ajm.2012.0008

[4] *A. Beauville, Quantum cohomology of complete intersections*, preprint.

[5] *X.Pan, Moduli Space of 2-Minimal-Dominant Rational Curves On Low Degree Complete Intersections* Preprint, 2012. http://www.math.columbia.edu/ pan/conics.pdf

[6] *X.Pan, Moduli Space of M-Minimal-Dominant Rational Curves On Low Degree Complete Intersections* Preprint, 2012. http://www.math.columbia.edu/ pan/mcurves.pdf


## Rational Connectedness of Moduli Space

- With the hypothesis as in Theorem 3, if $n + 3 - \sum_{i=1}^{c} d_i^2 > 0$. Then $\mathcal{F}$ is a Fano variety, hence, a rationally connected variety.
- With the hypothesis as in Theorem 3, if

$$m\{\sum_{i=1}^{c} \frac{d_i(d_i - 1)}{2} - 1\} + \sum_{i=1}^{c} d_i \leq n,$$

then $\mathcal{F}$ is rationally connected.

## Enumerative Geometry

- Let $X$ be a smooth complete intersection of degree $(d_1, \ldots, d_r)$ in $\mathbb{P}^{n+r}$, with $n = 2\sum(d_i - 1) - 1$. Then the number of conics in $X$ passing through 2 general points is $\frac{1}{2d}\prod_{i=1}^{r}(d_i!)^2$, where d is the degree of $X$.
- Let $X$ as above with $n = 3\sum(d_i - 1) - 3$. Then the number of twist cubics in $X$ passing through 3 general points is $\frac{1}{d^2}\prod_{i=1}^{r}(d_i!)^3$, where d is the degree of $X$.
- Let $X$ be as above with $n = 4\sum(d_i - 1) - 4$. Let $S$ be the set consisting of linking conics $C$ in $X$ which pass through 4 general points $p_1, p_2, p_3, p_4$ such that the points $p_1, p_2$ are on the same component of $C$ and $p_3, p_4$ are on the another component. The number of $S$ is

$$\frac{1}{d^3}\prod_{i=1}^{r}(d_i!)^4,$$

where d is the degree of $X$.

## Searching for a new 2-Fano variety

In the paper [3], C. Araujo and A. Castravet give a general formula of canonical bundle of the polarizing minimal families $H$ of rational curves for a projective unirule variety $X$, see [3]. They prove $H$ is Fano if $X$ is 2-Fano. This suggest that if $X$ is $n$-Fano then the moduli space of rational curves is $(n-1)$-Fano in some sense. In other words, there is a chance we may a new 2-Fano variety from the general fibers $\mathcal{F}$. But Theorem 2 and 3 give a negative answer to the expectation of finding a new 2-Fano variety.