Brill-Noether theory of tropical curves

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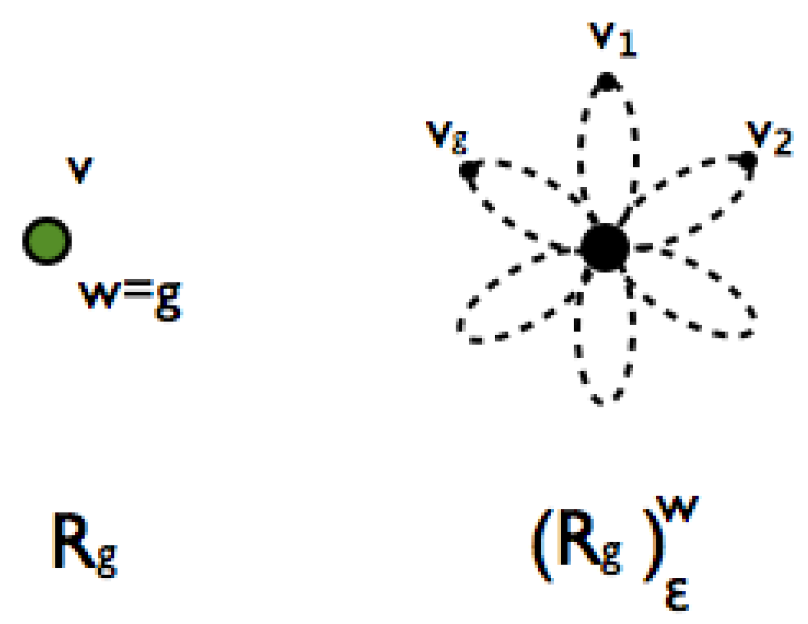
Tropical curves and their divisor theory

A **tropical curve** Γ is a triple (G, w, l), where (G, l) is a metric graph and w is a non-negative integer weight function on the vertices of G.

A **divisor** D is a finite combintaion of points of Γ. The **degree** of D is the sum its coefficients. A divisor is called **effective** if all of its coefficients are non-negative.

A **rational function** on Γ is a continuous piecewise-linear function. To a rational function f, we associate a divisor, denoted $\text{div}(f)$, whose value at every point p equals the sum of incoming slopes of f at p. Two divisors are called **linearly equivalent** if their difference is $\text{div}(f)$ for some f.

For a tropical curve Γ, let Γ_ϵ^w be the graph obtained by adding $w(p)$ loops of length ϵ at every point p. For example, if R_g is the curve consisting of a single vertex with weight g, then $(R_g)_\epsilon^w$ is the graph known as the "rose with g petals":



The **rank** of a divisor D is the largest integer r, such that for every effective divisor E of degree r which is supported on Γ_ϵ^w, $D - E$ is linearly equivalent to an effective divisor.

The **Brill-Noether rank** of a tropical curve, denoted w_d^r, is the largest integer ρ, such that every effective divisor of degree $r + \rho$ is contained in a divisor of degree d and rank at least r.

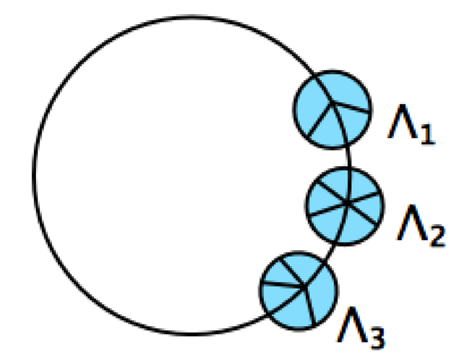
The moduli space of tropical curves

The **combinatorial type** of a tropical curve $\Gamma = (G, w, l)$ is (G, w). Fix a combinatorial type (G, w), and let $\sigma(G, w) = \mathbb{R}_{\geq 0}^{|E(G)|}$. Then each point $s = (s_1, ..., s_n)$ in the interior of $\sigma(G, w)$ corresponds to a tropical curve of type (G, w) with $l(e_i) = s_i$. Points on the boundary correspond to curves with an underlying graph obtained from G by contracting certain edges.

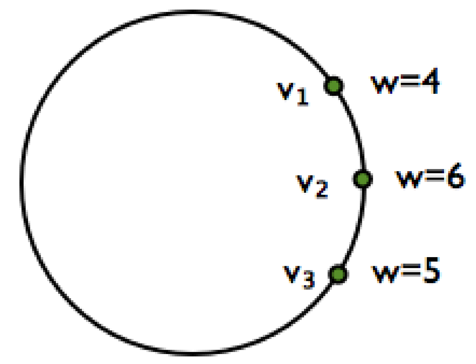
Following Brannetti, Melo, and Viviani, the moduli space of tropical curves is

$$M_g^{\text{tr}} = \coprod \sigma(G, w)/\sim$$

where two points are identified if they correspond to isomorphic curves. Consequently, when the length of a certain loop on a graph is contracted to zero, the resulting tropical curve is identified with the one having an extra weight on the vertex at the base of that loop. This implies that when any subgraph of genus h is contracted to a point, the resulting curve will be the weighted tropical curve with an added weight of h. For example, in the following tropical curve,



by decreasing the lengths of the edges of the subgraphs Λ_1, Λ_2 and Λ_3 to 0, we obtain vertices with weights 4,6,5:



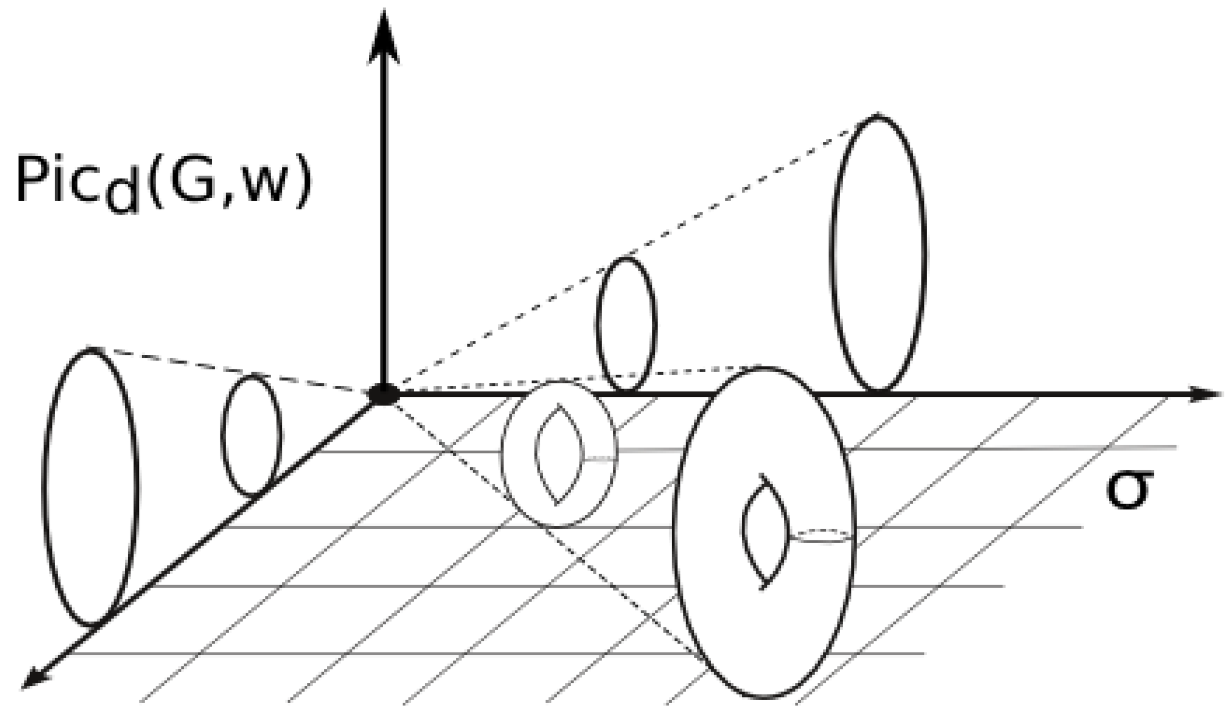
The universal Picard space

The Picard group, $\text{Pic}_d(\Gamma)$, of a tropical curve Γ is the set of divisor classes of degree d on Γ. As shown by Baker and Faber, $\text{Pic}_d(\Gamma)$ is naturally identified with a torus of rank g_0 where g_0 is the first Betti number of Γ. The Brill-Noether locus, $W_d^r(\Gamma)$ is the subset of divisor classes of rank at least r.

We define the **Universal Picard Space**, $\text{Pic}_d(G, w)$, as the space classifing divisors of degree d on all the tropical curves of combinatorial type (G, w). As a set,

$$\text{Pic}_d(G, w) = \coprod_{s \in \sigma} \{s\} \times \text{Pic}_d(\Gamma_s),$$

the disjoint union of the Picard groups of the different curves of type (G, w). We give this space the quotient topology making the natural map $\sigma \times (\Gamma_1)^d \to \text{Pic}_d(G, w)$ continuous.



The **The Universal Brill-Noether locus**, $W_D^r(G, w)$, is the subset of $\text{Pic}_d(G, w)$ consisting of classes of divisors of rank at least r.

Main results

Our main result is the following:

Theorem. *The universal Brill-Noether locus, $W_d^r(G, w)$, is closed in the universal Picard Space $\text{Pic}_d^r(G, w)$.*

In other words, whenever $(\Gamma_n, [D_n])$ is a sequence converging in $\text{Pic}_d^r(G, w)$ to $(\Gamma, [D])$, such that the rank of each $[D_n]$ is at least r, then the rank of $[D]$ is at least r as well.

As a corollary we obtain

Corollary. *The Brill-Noether rank is upper semi-contintuous on M_g^{tr}, the moduli space of curves of genus g.*

Moreover, we extend the Brill-Noether Specialization Lemma given by Lim, Payne and Potashnik to the setting of tropical curves:

Theorem. *Let X be a smooth projective curve over a discretely valued field with a regular semistable model whose special fiber has a weighted dual graph Γ. Then for every $d, r \in \mathbb{N}$, $\dim W_d^r(X) \leq w_d^r(\Gamma)$.*

References

Y. Len, *First steps in Brill-Noether theory of tropical curves*, 2012, preprint arXiv:1209.6309v1.

This poster and other material can be found on my website: `http://pantheon.yale.edu/~yl523/Yoav_Len.html`.