Tilt-stability on threefolds

Yogesh More, joint work with Jason Lo
SUNY Old Westbury, University of Missouri

Set-up

We study the construction introduced in [BMT]:

- X is a smooth projective 3-fold over $\mathbb{C}$.
- $\mathrm{Coh}(X)$ is the category of coherent sheaves on X.
- $D^b(X) = D^b(\mathrm{Coh}(X))$ is the bounded derived category of $\mathrm{Coh}(X)$.
- ω, B are two numerical equivalence classes of $\mathbb{Q}$-divisors on X, with ω an ample class.
- $\widetilde{ch}(E) = e^{-B}ch(E)$ is the twisted Chern character.
- Motivated by Π-stability in string theory, define a central charge $Z_{\omega,B} : D^b(X) \to \mathbb{C}$ by
$$Z_{\omega,B}(E) = -\int_X e^{-B-i\omega}\mathrm{ch}(E)$$
$$= (-\widetilde{ch}_3(E) + \frac{\omega^2}{2}\widetilde{ch}_1(E)) + i(\omega\widetilde{ch}_2(E) - \frac{\omega^3}{6}\widetilde{ch}_0(E))$$
- $\mu_{\omega,B}$-slope on $\mathrm{Coh}(X)$ is defined as follows. If $E \in \mathrm{Coh}(X)$ is a torsion sheaf, set $\mu_{\omega,B}(E) = +\infty$. Otherwise set
$$\mu_{\omega,B}(E) = \frac{\omega^2\widetilde{ch}_1(E)}{\widetilde{ch}_0(E)} = \frac{\omega^2(\mathrm{ch}_1(E) - B\mathrm{rk}(E))}{\mathrm{rk}(E)}.$$
- $\mathcal{T}_{\omega,B} \subset \mathrm{Coh}(X)$ is defined as the extension-closed subcategory generated by $\mu_{\omega,B}$-semistable sheaves of slope $\mu_{\omega,B} > 0$.
- $\mathcal{F}_{\omega,B} \subset \mathrm{Coh}(X)$ is defined as the extension-closed subcategory generated by $\mu_{\omega,B}$-semistable sheaves of slope $\mu_{\omega,B} \le 0$.
- The abelian category $\mathcal{B}_{\omega,B}$ is defined as the tilt of $\mathrm{Coh}(X)$ with respect to the torsion pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$:
$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle.$$
- For $E \in \mathcal{B}_{\omega,B}$, define its tilt-slope $\nu_{\omega,B}(E)$ as follows. If $\omega^2\widetilde{ch}_1(E) = 0$, then set $\nu_{\omega,B}(E) = +\infty$. Otherwise set
$$\nu_{\omega,B}(E) = \frac{\Im Z_{\omega,B}(E)}{\omega^2\widetilde{ch}_1(E)} = \frac{\omega\widetilde{ch}_2(E) - \frac{\omega^3}{6}\widetilde{ch}_0(E)}{\omega^2\widetilde{ch}_1(E)}.$$

Definition of tilt-stability

An object $E \in \mathcal{B}_{\omega,B}$ is defined to be **tilt-stable** or $\nu_{\omega,B}$-stable if, for any non-zero proper sub-object $F \subset E$ in $\mathcal{B}_{\omega,B}$, we have $\nu_{\omega,B}(F) < \nu_{\omega,B}(E/F)$.

Set-up continued

- $\mathcal{T}'_{\omega,B}$ is defined as the extension closed subcategory of $\mathcal{B}_{\omega,B}$ generated by $\nu_{\omega,B}$-stable objects $E \in \mathcal{B}_{\omega,B}$ of tilt-slope $\nu_{\omega,B}(E) > 0$.
- $\mathcal{F}'_{\omega,B}$ is defined as the extension closed subcategory of $\mathcal{B}_{\omega,B}$ generated by $\nu_{\omega,B}$-stable objects $E \in \mathcal{B}_{\omega,B}$ of tilt-slope $\nu_{\omega,B}(E) \le 0$.
- The abelian category $\mathcal{A}_{\omega,B}$ is defined as the tilt of $\mathcal{B}_{\omega,B}$ with respect to the torsion pair $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$:
$$\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle$$

Conjectural Bridgeland stability on threefolds, following [BMT]

In [BMT] it is proven that $\sigma = (Z_{\omega,B}, \mathcal{A}_{\omega,B})$ defines a Bridgeland stability condition on X if and only if the following conjecture holds:

[BMT, Conjecture 3.2.6]

Any tilt-stable object $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ satisfies
$$\widetilde{ch}_3(E) < \frac{\omega^2}{2}\widetilde{ch}_1(E).$$

In [BMT, Conjecture 1.3.1] they conjecture a stronger inequality
$$\widetilde{ch}_3(E) \le \frac{\omega^2}{18}\widetilde{ch}_1(E),$$
with the same hypothesis on E as above. In [BMT], [Mac] this stronger inequality is proven for $X = \mathbb{P}^3$.

Properties of tilt-stable objects

Proposition

If $E \in \mathcal{B}_{\omega,B}$ is tilt-semistable object with $\nu_{\omega,B}(E) < +\infty$, then $H^{-1}(E)$ is a reflexive sheaf.

Proposition

Let E be a line bundle with $\omega^2\widetilde{ch}_1(E) < 0$. Then there exists a constant $m_0 > 0$, depending only on $c_1(E)$, such that $E[1]$ is $\nu_{m\omega,B}$-stable whenever $m > m_0$.

Examples of tilt-stable objects

[BMT, Prop. 7.4.1] states that $\mu_{\omega,B}$-stable vector bundles E with vanishing discriminant $\overline{\Delta}_\omega(E) = 0$ are tilt-stable. We can construct new tilt-stable objects via:

Proposition

Suppose we have a short exact sequence in $\mathcal{B}_{\omega,B}$
$$0 \to E' \to E \to Q \to 0$$
where $Q \in \mathrm{Coh}^{\le 0}(X)$.

1. If E is tilt-stable, then E' is tilt-stable.
2. Assuming $\mathrm{Hom}(\mathrm{Coh}^{\le 0}(X), E) = 0$ and $\omega^2\widetilde{ch}_1(E) \ne 0$, if E' is tilt-stable then E is tilt-stable.
3. If E satisfies Conjecture 3.2.6 then E' also satisfies the same conjecture.

Proposition

Let $B = 0$. Suppose $\mathrm{Pic}(X)$ is generated by an ample line bundle L on X. Let $h = c_1(L)$, $D = h^3$, and $\omega = mh$ for some positive $m \in \mathbb{Q}$. Let $C \subset X$ be a curve in X, and let $d = h\cdot[C]$ be its degree. Let I_C be the ideal sheaf of $C \subset X$, and let
$$E := L^2 \otimes I_C.$$

1. If $\nu_{\omega,0}(E) = 0$ and $d < \frac{3}{2}D$, then E is $\nu_{\omega,0}$-stable.
2. If $-ch_3(\mathcal{O}_C) \le \frac{4}{3}d$ and $\nu_{\omega,0}(E) = 0$ then E satisfies [BMT, Conjecture 1.3.1].
3. IF $X \subset \mathbb{P}^4$ is a hypersurface of degree D, and $d \le D$, and $\nu_{\omega,0}(E) = 0$, then E satisfies [BMT, Conjecture 1.3.1].

References

[BMT] A. Bayer, E. Macri, and Y. Toda. Bridgeland stability conditions on threefolds I. arXiv:1103.5010v1 [math.AG].

[LM] J. Lo and Y. More. Some examples of tilt-stable objects on threefolds. arXiv:1209.2749 [math.AG].

[Mac] E. Macri. A generalized Bogomolov-Gieseker inequality for three dimensional projective space. arXiv:1207.4980v1 [math.AG].