# Tropical Complexes

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## Abstract

*Our goal is to compute information about algebraic varieties from semistable degenerations. In particular, we will work with the dual complex together with some additional intersection numbers, which we will call a* tropical complex. *A tropical complex extends the analogies between curves and graphs to higher dimensions.*

## Tropical complexes

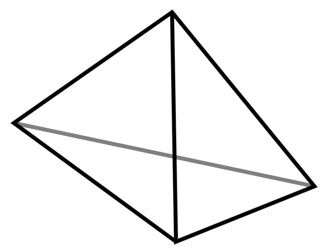
$R$: discrete valuation ring
$\mathfrak{X}$: regular scheme, flat and proper over $\operatorname{Spec} R$ whose special fiber is a reduced simple normal crossing divisor
$X$: generic fiber of $\mathfrak{X}$
$n$: dimension of $X$
$\Gamma$: dual complex of special fiber of $\mathfrak{X}$
$C_F$: component of special fiber corresponding to simplex $F$
$a(v, F)$: intersection number of $C_v$ with $C_F$ when $v$ is a vertex of an $(n-1)$-dimensional simplex $F$

The *dual complex* of a simple normal crossing divisor is the $\Delta$-complex which has a vertex for each irreducible component and a simplex for each component of the intersection between divisors. The *tropical complex* is the dual complex together with the integers $a(v, F)$.

**Example 1.** *Assume that $2$ is invertible in $R$ and let $\pi$ be a uniformizer of $R$. Let $\mathfrak{X}$ be a small resolution of*

$$V(xyzw - \pi(x^4 + y^4 + z^4 + w^4)) \subset \mathbb{P}^3_R$$

*The special fiber of $\mathfrak{X}$ consists of $4$ components and the dual complex is the boundary of a tetrahedron:*



*The integers $a(v, F)$ depend on the choice of small resolution. The symmetric choice leads to $a(v, F) = -1$ for every $v \in F$.*

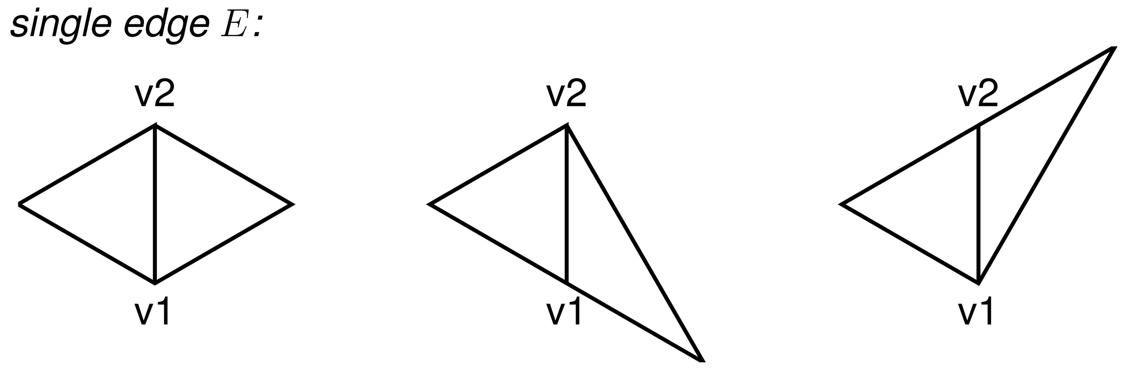
## Local embeddings

The integers $a(v, F)$ locally define balanced unimodular maps to a real vector space. Let $F$ be a $(n-1)$-dimensional simplex in a tropical complex $\Gamma$.

$N(F)$: subcomplex of all simplices containing $F$
$N(F)^o$: union of interiors of $F$ and of simplices containing $F$
$v_1, \ldots, v_n$: vertices of $F$
$w_1, \ldots, w_d$: vertices of $N(F)$ not in $F$
$V_F$: vector space $\mathbb{R}^{n+d} / \big(a(v_1, F), \ldots, a(v_n, F), 1, \ldots, 1\big)$
$\phi_F$: linear map $N(F) \to V_F$ sending $v_i$ and $w_j$ to images of $i$th and $(n+i)$th unit vectors respectively.

A continuous $\mathbb{R}$-valued function on an open subset $U \subset \Gamma$ is *linear* if on each $U \cap N(F)^o$ it is the composition of $\phi_F$ followed by an affine linear function with integral slopes. This defines a sheaf of linear functions on $\Gamma$.

**Example 2.** *Some embeddings of two triangles glued along a single edge $E$:*




$a_1 = a_2 = -1$ $\quad$ $a_1 = -2, a_2 = 0$ $\quad$ $a_1 = 0, a_2 = -2$

where $a_i$ is short for $a(v_i, E)$.

## Divisors and curves

The analogy to classical algebraic geometry is that a linear function is a non-vanishing regular function and a piecewise linear function is a rational function.

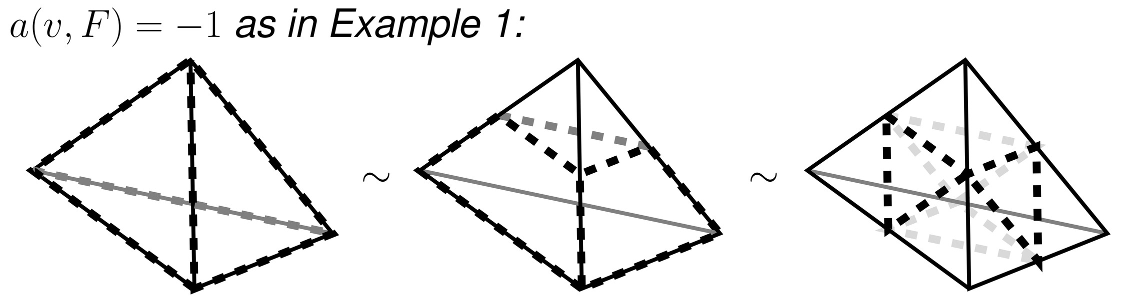
**Definition 3.** *A* PL function *on $\Gamma$ is a continuous function which is piecewise linear with integral slopes on each simplex.*

Given a PL function $f$, the complement of the open set where it is linear is a union of $(n-1)$-dimensional rational polyhedra.

**Definition 4.** *A* divisor *is a formal integral sum of $(n-1)$-dimensional polyhedra. A* Cartier divisor *is a formal integral sum of $(n-1)$-dimensional polyhedra which is locally the divisor of a PL function. A $\mathbb{Q}$-*Cartier divisor *is a divisor such that some multiple is Cartier.*

*Two divisors are* linearly equivalent *if their difference is the divisor of a PL function.*

**Example 5.** *Linearly equivalent divisors on a tetrahederon with $a(v, F) = -1$ as in Example 1:*



The group of Cartier divisors modulo linear equivalence, $\operatorname{Pic}(\Gamma)$, is isomorphic to the sheaf cohomology $H^1(\Gamma, \mathcal{A})$, where $\mathcal{A}$ is the sheaf of linear functions on $\Gamma$. Following Mikhalkin and Zharkov, this group fits into an "exponential sequence":

$$H^0(\Gamma, \mathcal{D}) \to H^1(\Gamma, \mathbb{R}) \to \operatorname{Pic}(\Gamma) \to H^1(\Gamma, \mathcal{D}) \to H^2(\Gamma, \mathbb{R}), \qquad (1)$$

where $\mathcal{D}$ is the quotient of $\mathcal{A}$ by the sheaf of locally constant functions. For a $1$-dimensional complex or a $2$-dimensional complex satisfying the hypotheses of of Theorem 14, the first map in (1) is an injection.

Let $C$ be a formal sum of edges in $\Gamma$ with rational slopes. For a PL function $f$ on $C$, we can define a formal sum of points:

$$\sum_{p \in C} \left( \sum_{E} (\text{outgoing slope of } f)(\text{multiplicity of } E \text{ in } C) \right) [p],$$

where $E$ ranges over edges of $C$ containing $p$. Curves are formal sums of edges which satisfy a balancing condition:

**Definition 6.** *We say that $C$ is a curve if for every linear function $f$ on an open subset $U \subset \Gamma$, the formal sum associated to $f|_C$ is trivial.*

## Intersection Numbers

**Definition 7.** *If $D$ is a $\mathbb{Q}$-Cartier divisor and $C$ a curve, then we define $D \cdot C$ to be the formal sum of points obtained from the restriction of a local defining equation of $D$ to $C$.*

**Proposition 8.** *This intersection product is well-defined. The total multiplicity of the points is invariant under linear equivalence of $D$.*

*If $n = 2$, then $C$ is a curve if and only if it is a $\mathbb{Q}$-Cartier divisor, and the intersection between two curves is symmetric.*

**Example 9.** *The self-intersection of the leftmost divisor from Example 5 is $4$ times the sum of the vertices. The self-intersection of the rightmost divisor is $2$ times the sum of the midpoints of the edges. In both cases, the total degree is $12$.*

## Equivalence to algebraic intersections

If $Y$ is a subvariety of dimension $k$ in $X$, then we define its tropicalization:

$$\operatorname{Trop}(Y) = \sum_{F \in \Gamma_k} (C_F \cdot \overline{Y})[F],$$

where $\overline{Y}$ is the closure of $Y$ in $\mathfrak{X}$.

**Theorem 10.** *Suppose that for each vertex $v$ in $\Gamma$, the vector spaces of numerical classes of curves and divisors on $C_v$ are spanned by $C_F$ and $C_E$ respectively as $F$ range over facets containing $v$ and $E$ ranges edges containing $v$.*

*If $D$ is a divisor on $X$ of $\mathfrak{X}$ and $C$ a curve, then $\operatorname{Trop}(D)$ and $\operatorname{Trop}(C)$ are a $\mathbb{Q}$-Cartier divisor and curve on $\Gamma$ respectively and the degree of $D \cdot C$ equals the degree of $\operatorname{Trop}(D) \cdot \operatorname{Trop}(C)$.*

## Dimension of global sections

**Definition 11.** *A point $x \in \Gamma$ is* rational *if its coordinates are rational when a containing simplex is identified with the standard unit simplex in $\mathbb{R}^k$.*

*We define $h^0(\Gamma, D)$ to be the cardinality of the smallest set of rational points $x_1, \ldots, x_k$ such that there is no effective divisor $D'$ linearly equivalent to $D$ such that the support of $D'$ contains $x_1, \ldots, x_k$. If there is no such set, then $h^0(\Gamma, D)$ is defined to be infinity.*

For $F$ a $k$-dimensional simplex of $\Gamma$, let $D_F \subset C_F$ be the sum of the $C_{F'}$ for $F'$ a $(k+1)$-dimensional simplex containing $F$.

**Theorem 12.** *Assume that for each vertex $v$ of $\Gamma$, $D_v$ is a big divisor on $C_v$, and that for each face $F$ of dimension $1 < k < n$, the difference $C_v \setminus D_v$ is quasi-affine.*

*For any divisor $D$ on $X$, we have the inequality:*

$$\dim H^0\big(X, \mathcal{O}(D)\big) \leq h^0\big(\Gamma, \operatorname{Trop}(D)\big)$$

## Abstract tropical complexes

We can axiomatize tropical complexes as follows:

**Definition 13.** *An $n$-dimensional* tropical complex *$\Gamma$ consists of the following data:*

- *A finite, connected $\Delta$-complex, whose maximal simplices have dimension $n$.*
- *Integers $a(v, F)$ for every pair of an $(n-1)$-dimensional simplex $F$ and a vertex $v \in F$.*

*which satisfy the following two conditions:*

- *For each facet $F$,*
$$\sum_{v \in F} a(v, F) = -\deg(F)$$
*where $\deg(F)$ is defined to be the number of $n$-dimensional simplices containing $F$.*
- *For each $(n-2)$-dimensional face $E$, we define $M_E$ to be the symmetric matrix whose rows and columsn are labelled by facets $F_i$ containing $E$ and whose $(i, j)$ entry is $a(F_i \setminus E, F_i)$ if $i = j$ and is the number of simplices containing both $F_i$ and $F_j$ otherwise. Here, $F_i \setminus E$ denotes the vertex of $F_i$ not containing $E$. We require that $M_E$ has exactly one positive eigenvalue for each $E$.*

For the rest of this section, $\Gamma$ will be $2$-dimensional

**Theorem 14** (Hodge index theorem)**.** *Suppose that $\Gamma$ is locally connected through codimension $1$. Then if $H$ and $D$ are divisors on $\Gamma$ such that $H^2 > 0$ and $H \cdot D = 0$, then $D^2 \leq 0$.*

**Conjecture 15** (Full Hodge index theorem)**.** *If $\Gamma$ is locally connected through codimension $1$, then $\Gamma$ has a divisor $H$ such that $H^2 > 0$.*

We define the *canonical divisor* to be:

$$K_\Gamma = \sum_{F \in \Gamma_{n-1}} (\deg(F) - 2)[F]$$

**Conjecture 16** (Riemann-Roch)**.** *Let $D$ be a $\mathbb{Q}$-Cartier divisor on a tropical surface $\Gamma$ and assume that $K_\Gamma$ is a $\mathbb{Q}$-Cartier divisor. Then*

$$h^0(\Gamma, D) + h^0(\Gamma, K_\Gamma - D) \geq \frac{D \cdot (D - K_\Gamma)}{2} + \chi(\Gamma),$$

*where $\chi(\Gamma)$ is the topological Euler characteristic of $\Gamma$ as a $\Delta$-complex.*

If $K_\Gamma$ is $\mathbb{Q}$-Cartier, define the second Chern class to be the formal sum of points

$$c_2(\Gamma) = -K_\Gamma^2 + \sum_{v \in \Gamma_0} \Bigg( 12 + 3\#\{\text{triangles} \ni v\} - \sum_{E = (v,w):\ \text{edge}} (6 + a(w, E)) \Bigg)[v]$$

In this case, we have Noether's formula:

$$12\chi(\Gamma) = K_\Gamma^2 + c_2(\Gamma)$$