Sharp Slope Bounds for Sweeping Families of Trigonal Curves

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Introduction

The log minimal model program for $\overline{\mathcal{M}}_g$ seeks functorial interpretations of the various log canonical models

$$\overline{\mathcal{M}}_g(\alpha) = \operatorname{Proj} \bigoplus_{n \geq 0} H^0(\overline{\mathcal{M}}_g, n(K_{\overline{\mathcal{M}}_g} + \alpha\delta)),$$

and the rational maps between them. As a first step, one must understand the stable base loci of the linear series $|K_{\overline{\mathcal{M}}_g} + \alpha\delta|$. We expect these base loci to consist of loci of curves with exceptional properties (e.g. existence of special linear series). As a first attempt, we may ask the reverse question.

Question

Consider $X \subset \overline{\mathcal{M}}_g$ parametrizing curves with some intrinsic properties (e.g. the locus of hyperelliptic curves). For which α is it covered by $(K_{\overline{\mathcal{M}}_g} + \alpha\delta)$-negative curves? Equivalently, what is the largest s such that it is covered by curves of slope $\delta/\lambda = s$?

Known results and main theorem

We know the answer to the question for the locus of hyperelliptic curves.

Theorem (Cornalba,Harris [3])

The locus of hyperelliptic curves is swept by curves of slope $8 + 4/g$, and any curve sweeping this locus must have at most this slope.

We know the answer for the locus of trigonal curves of even genus.

Theorem (Barja, et al. [2], [5], [6])

If g is even, the locus of trigonal curves is swept by curves of slope $7 + 6/g$, and any curve sweeping this locus must have at most this slope.

Our result settles the question for trigonal curves of all genera.

Theorem

Let $g \geq 4$. Denote by $\overline{\mathcal{T}}_g$ the closure in $\overline{\mathcal{M}}_g$ of the locus of smooth trigonal curves. Set

$$s_g = \begin{cases} 7 + 6/g & \text{if } g \text{ is even,} \\ 7 + 20/(3g+1) & \text{if } g \text{ is odd.} \end{cases}$$

Then $\overline{\mathcal{T}}_g$ is swept by curves of slope s_g and any curve sweeping $\overline{\mathcal{T}}_g$ must have at least this slope.

The idea behind the proof is classical, but the methods are modern. The proof consists of two parts

1. Construct sweeping curves that achieve the bound s_g.
2. Construct an effective divisor $D \subset \overline{\mathcal{T}}_g$ of class

$$[D] \sim s_g\lambda - \delta - \text{Effective combination of higher boundary},$$

ensuring that s_g is sharp.

Structure of triple covers

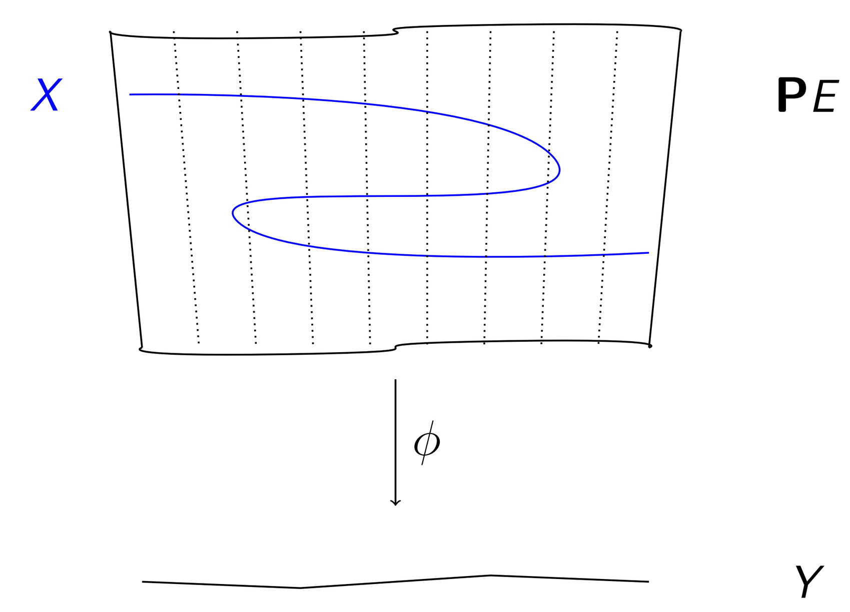
Let $\phi\colon X \to Y$ be a (Gorenstein) triple cover. We have the split sequence

$$0 \to O_Y \to \phi_* O_X \to E^\vee \to 0,$$

where E is a vector bundle of rank two on Y. Then X can be embedded in $\mathbf{P}E$ as a divisor of class

$$O_{\mathbf{P}E}(X) = O_{\mathbf{P}E}(3) \otimes \det E^\vee.$$

Conversely, a generic such divisor gives a triple cover.

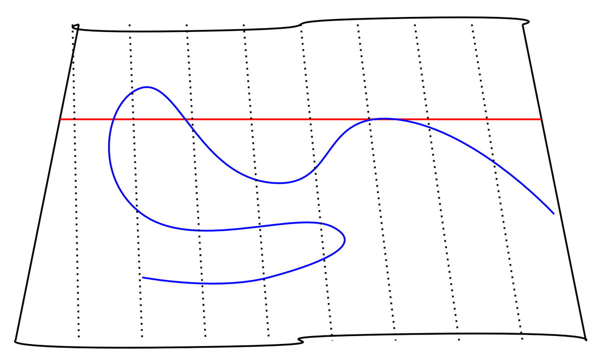


Sweeping curves

1. Even genus: Let $g = 2n - 2$ and take a pencil of $(3, n)$ curves on $\mathbf{P}^1 \times \mathbf{P}^1$. Or in terms of the description above, take
$$Y = \mathbf{P}^1 \times \mathbf{P}^1 \text{ and } E = O_Y(n\sigma + F) \oplus O_Y(n\sigma + F),$$
where F is a fiber and σ a section of $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. This gives sweeping curves of slope $7 + 6/g$.
2. Odd genus: Let $g = 2n - 1$. Take
$$Y = \mathbf{P}^1 \times \mathbf{P}^1 \text{ and } E = O_Y(n\sigma + F) \oplus O_Y((n+1)\sigma + 2F),$$
where F is a fiber and σ a section of $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. This gives sweeping curves of slope $7 + 20/(3g+1)$.

Extremal effective divisors

We construct effective divisors to prove that the above construction is sharp. Indeed, any sweeping curve must intersect these divisors non-negatively, which gives a bound on its slope. By looking at the expression of these divisors in terms of λ, δ and other boundary divisors, we see that this bound is precisely s_g.

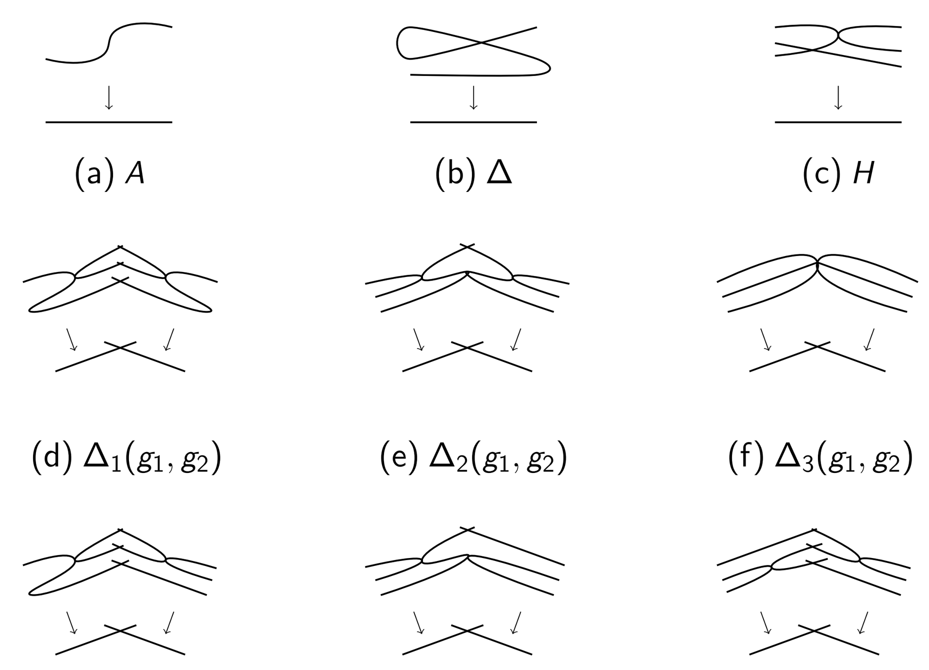


The divisors are defined as closures in a suitable compactification of the following loci:

1. Even genus: Let μ be the locus of trigonal curves whose E is *unbalanced*.
2. Odd genus: Let τ be the locus of trigonal curves that embed $\mathbf{F}_1$ and are tangent to the diretrix.

A better compactification

We use (a slight variant of) the admissible cover compactification of the space of trigonal curves, denoted by $\overline{\mathcal{H}}^3_g$. An open locus in $\overline{\mathcal{H}}^3_g$ consists of simply branched trigonal curves. The boundary consists of the following divisors:



(a) A (b) Δ (c) H

(d) $\Delta_1(g_1, g_2)$ (e) $\Delta_2(g_1, g_2)$ (f) $\Delta_3(g_1, g_2)$

(g) $\Delta_4(g_1, g_2)$ (h) $\Delta_5(g_1, g_2)$ (i) $\Delta_6(g_1, g_2)$

Theorem (Divisor class computations)

In the rational Picard group of $\overline{\mathcal{H}}^3_g$ for even g, we have

$$2(g-3)[\overline{\mu}] = (7g+6)\lambda - g\delta - cH - \sum c_i(g_1, g_2)\Delta_i(g_1, g_2),$$

and for odd g, we have

$$2[\overline{\tau}] = (21g+27)\lambda - (3g+1)\delta - cH - \sum c_i(g_1, g_2)\Delta_i(g_1, g_2),$$

where c and $c_i(g_1, g_2)$ are (explicitly given) non-negative numbers. As a result, the sweeping curves we have constructed are of maximum slope.

The proof involves test-curve calculations using orbinodal covers of Abramovich–Corti–Vistoli [1] and Grothendieck–Riemann–Roch for orbifolds [4]. Orbinodal curves allow us to simplify the approach essentially laid out by Stankova in [6], and will be useful in finding slope bounds of families of higher degree covers.

References

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