A Divisor with Non-Closed Restricted Base Locus

John Lesieutre

Department of Mathematics, Massachusetts Institute of Technology

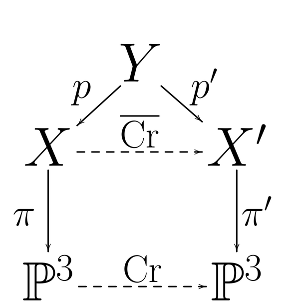
The restricted base locus

- The *stable base locus* of a $\mathbb{Q}$-divisor D is
$$\mathbf{B}(D) = \bigcap_{n\geq 1} \mathrm{Bs}(nD),$$
where Bs is the usual base locus.
- If D is an $\mathbb{R}$-divisor, the *restricted base locus* and *augmented base locus* are [3]
$$\mathbf{B}_-(D) = \bigcup_{\substack{A \text{ ample} \\ D+A\ \mathbb{Q}\text{-Cartier}}} \mathbf{B}(D+A),$$
$$\mathbf{B}_+(D) = \bigcap_{\substack{A \text{ ample} \\ D-A\ \mathbb{Q}\text{-Cartier}}} \mathbf{B}(D-A).$$
- $\mathbf{B}_\pm(D)$ depends only on the numerical class of D.
- If $D \cdot C < 0$, then $C \subseteq \mathbf{B}_-(D)$.
- $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef.
- While $\mathbf{B}_+(D)$ is clearly closed, we will see that $\mathbf{B}_-(D)$ is not in general.

The cubic Cremona transformation

The cubic Cremona transformation centered at four non-coplanar points $p_1, \ldots, p_4$ is the rational map $\mathrm{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by
$$[X_0, X_1, X_2, X_3] \mapsto [X_0^{-1}, X_1^{-1}, X_2^{-1}, X_3^{-1}],$$
where the coordinates are chosen so each point has a single nonzero coordinate.

- Cr has a resolution



where π and π' are the blow-up of $\mathbb{P}^3$ at four points, and $\overline{\mathrm{Cr}}$ is the flop of the strict transforms of the six lines through two points.
- The map $\overline{\mathrm{Cr}} : X \dashrightarrow X'$ is an isomorphism in codimension 1, and strict transform induces $M : N^1(X) \to N^1(X')$ given by
$$M = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{pmatrix},$$
with respect to the basis $\pi^*H, E_1, \ldots, E_4$.

Composing Cremona transformations

Let X be the blow-up of $\mathbb{P}^3$ at 9 very general points.

- Repeat the following two steps.
  1. Make a Cremona transformation centered at $p_6, \ldots, p_9$.
  2. Cyclically reindex the points with these four first.
- If D is a divisor, its strict transform under n iterations of the above procedure lies in the class $M_\sigma^n([D])$, where $M_\sigma : N^1(X) \to N^1(X)$ is given by
$$M_\sigma = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_5 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_\sigma \end{array}\right),$$
with Π_σ the matrix for the permutation (678912345).
- If C is a curve disjoint from the indeterminacy locus of each of the Cremona transformations, its intersections with divisors are unchanged under strict transform.
- The transform of C has class $N_\sigma^n([C])$, where $M_\sigma^{\mathrm{T}} I_{1,9} N_\sigma = I_{1,9}$.

Strict transforms of a line

- If C_0 is the line between p_1 and p_2, its strict transforms C_n are disjoint from the indeterminacy loci. [4]

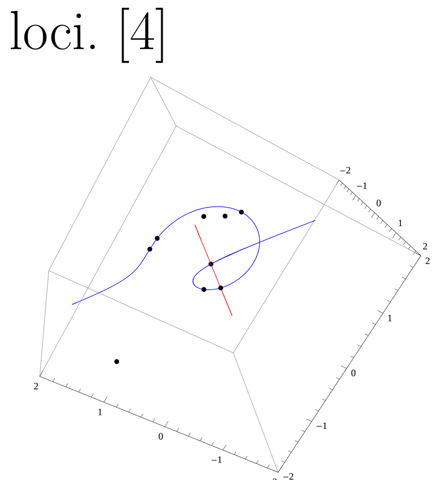


Figure 1: A line and the strict transform of a line

- Thus C_n is a curve of class $N_\sigma^n([C_0])$.
- In fact $\bigcup_{n\geq 0} C_n$ is Zariski dense.

n	δ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0	1	1	1	0	0	0	0	0	0	0
1	3	1	1	1	1	1	1	0	0	0
2	7	3	2	2	2	1	1	1	1	1
3	13	4	4	4	4	3	2	2	2	1
4	25	8	8	8	7	4	4	4	4	3

Table 1: The first few classes $[C_n] = \delta h - \sum \mu_i e_i$.

The main eigenvector of M_σ

- M_σ has a unique eigenvalue $\lambda \approx 1.800$ of norm greater than 1, with associated eigenvector
$$D_\lambda \approx (1; -0.640, -0.634, -0.615, -0.554, -0.355, -0.352, -0.341, -0.307, -0.197).$$
- Restricted base loci of eigenvectors more generally are studied by Bayraktar. [1]
- D_λ spans an extremal ray on both $\overline{\mathrm{Mov}}(X)$ and $\overline{\mathrm{Eff}}(X)$.
- Observe that $D_\lambda \cdot C_0 \approx -0.274 < 0$, and so
$$D_\lambda \cdot C_n = (\lambda^{-n} M_\sigma^n D_\lambda) \cdot (N_\sigma^n C_0) = \lambda^{-n} D_\lambda \cdot C_0 < 0.$$
- Since $D_\lambda \in \overline{\mathrm{Mov}}(X)$, $\mathbf{B}_-(D_\lambda)$ has no codimension-1 components.
- $\mathbf{B}_-(D_\lambda)$ is a countable union of curves.

Zariski decomposition

- Let D be a pseudoeffective $\mathbb{R}$-divisor on a smooth projective surface X. There exists an effective divisor $N = \sum_i a_i N_i$ such that $P = D - N$ is nef, $(N_i \cdot N_j)$ is negative definite, and $P \cdot N_i = 0$.
- Various ways to extend this to higher dimensions, surveyed in [6].
- Can we find $f : Y \to X$ and $f^*D \sim P + N$ with:
  1. CKM: the maps $H^0(Y, \mathcal{O}_Y(\lfloor mP \rfloor)) \to H^0(Y, \mathcal{O}_Y(\lfloor mf^*D \rfloor))$ are all isomorphisms.
  2. Fujita: if $g : Y' \to Y$ is birational, and $P' \leq g^*f^*D$ is nef, then $P' \leq g^*P$.
  3. Nakayama: $P = P_\sigma(f^*D)$ is the positive part of the divisorial Zariski decomposition.
- Say D admits a *weak Zariski decomposition* if we can write $f^*D \equiv_{\mathrm{num}} P + N$, with P nef and N effective.
- Birkar shows that this has implications for MMP: assuming LMMP in dimension $n-1$, a pair (X, Δ) has a minimal model if $K_X + \Delta$ admits a weak Zariski decomposition. [2]
- Question: does every pseudoeffective $\mathbb{R}$-divisor admit a weak Zariski decomposition?
- The above observations imply that D_λ does not.

Zariski decomposition of D_λ

- The following conditions are sufficient to guarantee that a divisor D does not admit a weak Zariski decomposition.
  1. $[D]$ spans an extremal ray on $\overline{\mathrm{Eff}}(X)$.
  2. D is not numerically equivalent to any effective divisor.
  3. There does not exist birational $f : Y \to X$ and a nef $\mathbb{R}$-divisor P on Y such that $f_*P = D$.
- The last of these follows for $D = D_\lambda$ from the fact that D_λ is negative on infinitely many curves.
- We can also produce a big $\mathbb{R}$-divisor which has no decomposition in any the three above senses (an earlier example is due to Nakayama [5]).
- Let CX be the projective cone over X and $Y \cong \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(1))$ the blow-up at the cone point, with $p : Y \to CX$ and $q : Y \to X$ the obvious maps.
- Set $D'_\lambda = p^*H + q^*D_\lambda$, where H is a large ample divisor on CX supported off of the cone point.
- This is big, but negative on the curves C_n in the exceptional divisor of p, and $\mathbf{B}_-(D'_\lambda)$ is not closed.

References

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