Allen Knutson. 1

Frobenius Spittings and Degeneration

- **Def** $\mathcal{Y}$ a stratification of $\mathbb{A}^n_{\mathbb{F}}$

$$\mathcal{Y} = \{ \underbrace{Y \subseteq \mathbb{A}^n_{\mathbb{F}}}_{\text{closed subvarieties}} \} \qquad \forall Y_1, Y_2 \in \mathcal{Y} \quad Y_1 \cap Y_2 = \bigcup_{\substack{Z \in \mathcal{Y} \\ Z \subseteq Y_1 \cap Y_2}} Z$$

- **Def** Strat generated by a hypersurface. $\{f=0\}$ by intersect, decompose, and repeat.

Ex $f = b(ac-b)$ $\quad a \gg b \gg c$

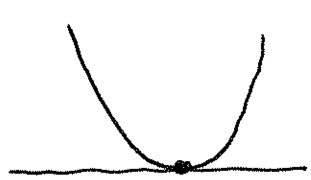
POSET of strata

$\mathcal{Y}$

$$\begin{tikzcd}
 & \mathbb{A}^3 \arrow[dl, dash] \arrow[dr, dash] & \\
b=0 \arrow[d, dash] \arrow[drr, dash] & & ac=b \arrow[d, dash] \arrow[dll, dash] \\
b=a=0 \arrow[dr, dash] & & b=c=0 \arrow[dl, dash] \\
 & a=b=c=0 &
\end{tikzcd}$$

Good set th.
Good sch. th.

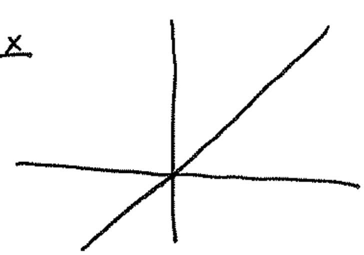
Ex $f = y(y-x^2)$



Good set th

Not good sch th.

Ex



Good both set and sch th.

▶ **Thm1** Let f be deg n in $\mathbb{F}[x_1, \ldots, x_n]$ $\quad n = n$.

Then $\forall\, Y_1, Y_2 \in \mathcal{Y}_f$, $\quad Y_1 \cap Y_2$ is reduced if either

① init $f = \prod_{i=1}^{n} x_i$ $\quad\boxed{\begin{array}{l}\text{[Ambro] char } 0\\ \text{[LMP] introduced it in char } p\end{array}}$

② $\mathbb{F} = \mathbb{F}_p$, and $\#\{f \neq 0\} \not\equiv 0 \mod p$.

and ① $\Rightarrow$ ② if $\mathbb{F} = \mathbb{F}_p$

and both $\Rightarrow$ $\left(\bigcup_S Y_s\right) \cap \left(\bigcup_T Y_t\right)$ reduced.

<u>More good f</u>:

$$M = \underbrace{\begin{bmatrix} \star & & \cdot^{\cdot^{\cdot}} 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}}_{N} \qquad f = \prod_{i=1}^{N-1} \det{}^{i}\left[\begin{bmatrix} \ {}^{i} \ \end{bmatrix} \quad M\right]$$

$$N = 3: \quad \begin{bmatrix} b & a & 1 \\ c & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathcal{Y}_f \simeq S_N.$$

$$\forall f \in S_N, \quad \overline{X_\pi} := \begin{bmatrix} \diagdown & 0 \\ \star & \diagdown \end{bmatrix} \pi \begin{bmatrix} \diagdown & \star \\ 0 & \diagdown \end{bmatrix} \subseteq M_N$$

matrix schubert variety.

Thm 2 Assume ①. Then

a) Each init Y is reduced, for $Y \in \mathcal{Y}_F$. They are also union of coord subspaces of $\mathbb{A}_F$.

b) Same for any $\operatorname{init}\left(\bigcup_s Y_s\right)$.

c) If $Y = \bigcap_{S \leq Y} Z$, then $\operatorname{init} Y = \bigcap \operatorname{init} Z$

d) $\exists$ well-defined map: $\left\{\begin{matrix}\text{faces of} \\ \Delta_{n-1}\end{matrix}\right\} \xrightarrow{\pi} \mathcal{Y}_f$

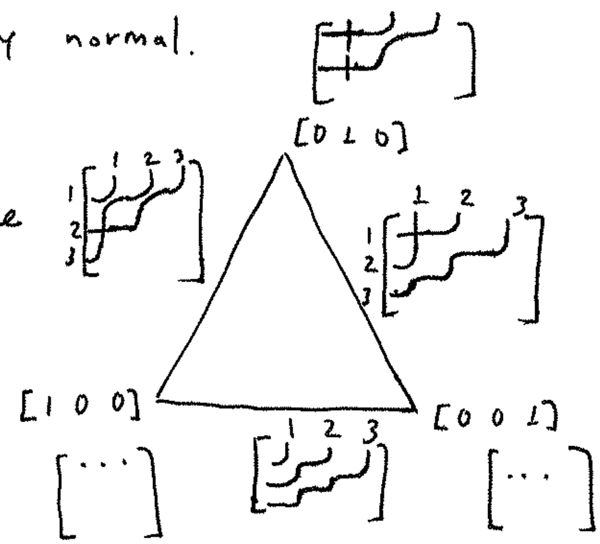
$F \subseteq \{1, \dots, n\} \longmapsto \min\{Y : \operatorname{init} Y \supseteq \mathbb{A}^F\}$

e) If $\overline{\pi^{-1}(Y)} \simeq$ ball $\therefore$ Y C-M.

If $\pi^{-1}(Y) =$ interior $\therefore$ Y normal.

<u>Example</u>: Pipe dreams: $+$ means 0, $\rfloor\!\ulcorner$ means free

$$\begin{bmatrix} b & a & 1 \\ c & 1 & \\ 1 & & \end{bmatrix}$$



	G	B	T	W	B_-
eg.	$Gl(n)$	$\begin{bmatrix} 0 & \ast \end{bmatrix}$	$\begin{bmatrix} \diagdown & 0 \\ 0 & \end{bmatrix}$	S_n	$\begin{bmatrix} \ast & 0 \end{bmatrix}$

Thm 3 let $X_\circ^{w\in W} := BwB/_B \subseteq G/B$ opposite Bruhat cell.
be stratified by its intersections w/ $\{ B_- vB/_B =: X_v^\circ \}$

Then for each reduced word Q in generators of W (gens. $(i \leftrightarrow i+1)$, $i = 1, \ldots, n-1$)
w/ product $= w$, $\exists$ iso $\mathbb{A}^{|Q|} \simeq X_\circ^w$.

$\exists f$ on $\mathbb{A}^{|Q|}$ satisfying:
1. $\operatorname{init} f = \prod_{i=1}^{n} x_i$

with $\mathcal{Y}_f$ = that stratification
and e) holds.

$$\Pi : \left\{ \begin{matrix} \text{subwords} \\ \text{of } Q \end{matrix} \right\} \longrightarrow \begin{matrix} \text{Bruhat} \\ \text{interval} \\ [1, w] \end{matrix}$$

$S \longmapsto$ greedy/nil Hecke / Demazure product.

Ambient Vectorspace	Stratification	$X_\circ^w$
M_N matrices	Matrix Schub. vars. $(\subseteq)$	$W = S_{2N}$ (Fulton 92') $w = N+1 \cdots 2N\ 1 \cdots N$
$\prod_{i=1}^{m-1} \operatorname{Hom}(V_i, V_{i+1})$	Equivalence of repr. $\prod GL(V_i)$	[Zelevinski, L-M].
Open patch on $Gr_k(\mathbb{A}^n)$	Lusztig, Postnikov, et al.	$W = \hat{S}_m$ [Snider] 10'.
Open patch on $\overline{G}$	$B \times B$ -orbits	W = nasty K-M weyl gp. [X. He, -J. Lu, K].

- R reduced $\iff \forall n > 1 \quad \forall r \quad r^n = 0 \Rightarrow r = 0$

  $\iff \exists n > 1, \quad \forall r \quad$ "$\ker(r \longrightarrow r^n) = 0$".

  This "ker" makes sense if $n = \operatorname{char} \mathbb{F} = p$.

- Def A Frobenious splitting φ of $R/\mathbb{F}_p$ is φ s.t.

  ① $\varphi(a+b) = \varphi(a) + \varphi(b)$

  ② $\varphi(a^p b) = a\,\varphi(b)$

  ③ $\varphi(1) = 1$.

  $I \subseteq R$ compatibly split if $\varphi(I) \subseteq I$.

[Rmk:
 . Being radical is an open condition in ideals.
 . Compatibley split is open condition in families of ideals andsplittings].