JK

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Local Version of Kawamata - Viehweg Vanishing theorem

- K.V. vanishing
  $\Delta = \sum d_i D_i \quad 0 \le d_i \le 1. \quad$ s.t. (X, Δ) is dlt.

$$L \sim_{\mathbb{Q}} K_X + M + \Delta \quad \Rightarrow \quad H^i(X, L) = 0 \quad i > 0.$$

- Thm (X, Δ) dlt. $p \in X$. D eff $\mathbb{Z}$-div.

s.t. $D \sim_{\mathbb{Q}} \underbrace{\Delta'}_{\text{eff}} \le \Delta$.

$$\Rightarrow \quad H^i_p(X, \mathcal{O}_X(-D)) = 0 \quad \text{for } i < \dim X.$$

equiv. $\mathcal{O}_X(-D)$ is CM.

- **Example**: $X = (xy - uv = 0) \subseteq \mathbb{A}^4$

$A = (x = u = 0) \qquad B = (x = v = 0)$

$\mathcal{O}_X(nA + mB) \qquad$ CM $\Leftrightarrow$ $|n - m| \le 1$.

- Corollaries: ① (X, Δ) dlt $\Rightarrow$ $\mathcal{O}_X$ CM $\quad (D = 0) \quad$ Elkik.

② Fujino $\mathcal{O}_X(-\lfloor \Delta \rfloor)$ CM $\quad D = \lfloor \Delta \rfloor = \Delta'$.

Thm 2 (X, Δ) l.c. $D \sim_{\mathbb{Q}} \Delta' \leq \Delta$

$\mathcal{O}_X(-D)$ if $p \in X$, $\operatorname{codim}(p, X) \geq 3$ p is not a l.c. center.

$\Rightarrow \operatorname{depth}_p \mathcal{O}_X(-D) \geq 3$.

(continuation of corollaries).

3) (X, Δ) l.c. $K_X + \Delta \sim_{\mathbb{Q}} 0$ $-K_X \sim_{\mathbb{Q}} \Delta$.

$\mathcal{O}_X(-(-K_X)) = \omega_X$ has depth ≥ 3

X l.c. fibers l.c.

$(X, \Delta) \to B$ ω_X has depth ≥ 3 at all closed pts.

$\Rightarrow \omega_X|_{X_p} \xrightarrow{\sim} \omega_{X_p}$

then these are a flat family ~~for~~ a family. l.c. pairs.

4) $\Delta = 0$ $(X, 0) \to B$ K_X is rel ample.

$$\underbrace{R(X_b, K_{X_b})}_{\Rightarrow \text{ for a flat family.}} = \sum_{m=0}^{\infty} H^0(X_b, mK_{X_b}).$$

(In Siu's thm he has global problems, here have local problems).

(X, Δ).

$$R(X, K_X+\Delta) = \sum_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m K_X + \lfloor m\Delta \rfloor)).$$

$$mK_X + \lfloor m\Delta \rfloor \sim_{\mathbb{Q}} m(K_X+\Delta) - (m\Delta - \lfloor m\Delta \rfloor)$$

$$m\Delta - \lfloor m\Delta \rfloor \overset{?}{\leq} \Delta$$

This condition holds for standard coeffs

i.e. $\Delta = \sum (1 - \frac{1}{m_i}) D_i$

5 (X, Δ) standard coeffs

$$\begin{array}{c}(X,\Delta)\\ \downarrow\\ B\end{array} \Rightarrow R(X_b, K_{X_b} + \Delta|_b) \text{ form a flat family.}$$

Example (where flatness fails).

$\mathbb{P}^2 \xrightarrow{\mathcal{O}(2)} \mathbb{P}^5$. $\quad \mathbb{P}^5 \xleftarrow{\mathcal{O}(1,2)} \mathbb{P}^1 \times \mathbb{P}^1$

$m = br + a$.

P 

Q $(Q, \frac{1}{r} D_Q)$.

$(P, \frac{1}{r} D_P)$

$(S, \frac{1}{r} D_S)$

"have common degeneration S".

$$\chi(P, mK_P + \lfloor \tfrac{m}{r} D_P \rfloor) = \binom{2n-2m-2}{2} - a(2n-2m-2) + \binom{a}{2}$$

$$\shortparallel$$

$$\chi(S, mK_S + \lfloor \tfrac{m}{r} D_S \rfloor) = \cdots$$

$$\chi(Q, mK_Q + \lfloor \tfrac{m}{r} D_Q \rfloor) = \cdots$$

$$\boxed{\text{Question: What if } \Delta = \sum a_i D_i,\ a_i \geq 1/2\ ?}$$

Ideas in the proofs:

"method of two spectral sequences".

$f : Y \longrightarrow X$

$f^{-1}(P) = W \longrightarrow P$

$F \longrightarrow F'$.

$$\begin{tikzcd} H^i_p(X, R^j f_* F') \arrow[r, Rightarrow] & H^{i+j}(Y, F') \\ H^i_p(X, R^j f_* F) \arrow[r, Rightarrow] \arrow[u] & H^{i+j}_W(Y, F) \arrow[u] \end{tikzcd}$$

Assume:

$$\begin{tikzcd} H^i_p(X, f_* F') \arrow[r, "\alpha_i", "\simeq"'] & H_W(Y, F') \\ H^i_p(X, f_* F) \arrow[r] \arrow[u, "\simeq"] & H_W(Y, F) \arrow[u] \end{tikzcd}$$

Lemma: Assume

1) $f_* F' = f_* F$.

2) $H^0_*(Y, F) = 0$

3) α_i's $\simeq$.

Prove

(X, Δ) flat $\Rightarrow$ $\mathcal{O}_X$ CM

1) ✓

$F = \mathcal{O}_Y$, $F' = \mathcal{O}_Y(B)$ $\quad B$ f-exc.

$$H^i_W(Y, \mathcal{O}_Y) = \varinjlim \mathcal{E}xt^i_Y(\mathcal{O}_Y / I_Y^m, \mathcal{O}_Y) \overset{\text{dual}}{\sim} \varprojlim H^{n-i}(Y, \omega_Y \otimes \mathcal{O}_Y / I_Y^m)$$

2) ✓

α_i are $\simeq$ if $\quad R^j f_* \mathcal{O}_Y(B) = 0 \quad j > 0$

$$B \sim_{\mathbb{Q}} K_Y + (\delta\text{-nef}) + \Delta_Y$$

$$K_Y \sim \delta^*(K_X + \Delta) + A$$

$A = \Sigma a_i A_i$

$a_i > -1 \ \forall i$

(because dlt).

$$K_Y - \underbrace{\delta^*(K_X + \Delta)}_{f\text{-nef}} + \Delta_Y \sim \underline{\underline{B}}$$