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▶ Pretalk: **Local cohomology.** R. Hartshorne, A. Grothendieck.

X scheme, S closed subscheme.

$\Gamma_Z(X, F)$ = those global sections whose support lies in Z.

- Examples: $X = \mathbb{A}^n$, $Z = \{0\}$

  ① $\Gamma_Z(X, \mathcal{O}_X) = 0$

  ② $\Gamma_Z(X, \mathbb{C}_0) = \mathbb{C}_0$. $\quad \mathbb{C}_0$ skyscraper sh.

- Def local cohomology: Right derived functors of Γ_Z. $\quad H^i_Z(X, F)$.

  1) $\Gamma_Z(X, F) \neq 0 \overset{X \text{ affine}}{\Longleftrightarrow} \exists\ \mathcal{O}_Z \longrightarrow F$.

  2) $\Gamma_Z(X, F) = \varinjlim \mathcal{H}om_X(\mathcal{O}_X/I_Z^m, F)$

▶ **Thm** $H^i_Z(X, F) \simeq \varinjlim \mathcal{E}xt^i_X(\mathcal{O}_X/I_Z^n, F)$

so $H^i_Z(X, F)$ are supported on Z.

$$0 \to \Gamma_Z(X, F) \to \Gamma(X, F) \to \Gamma(X \setminus Z, F|_{X\setminus Z}) \to$$
$$\to H^1_Z(X, F) \to H^1(X, F) \to H^1(X \setminus Z, F|_{X \setminus Z}) \to \cdots$$

$X = \mathbb{A}^1$ $\quad Z = p.$

$$0 \to H^0_p(\mathbb{A}^1, \mathcal{O}) \to H^0(\mathbb{A}^1, \mathcal{O}) \to H^0(\mathbb{A}^1 - 0, \mathcal{O}) \to H^1_p(\mathbb{A}^1, \mathcal{O}) \to 0 \cdots$$

$$H^1_p(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = \langle x^{-1}, x^{-2}, x^{-3}, \dots \rangle \qquad k[x]\text{-module.}$$

▶ **Thm**: $p \in X^n$, F coh on X. F is CM at $p \Leftrightarrow H^i_p(X, F) = 0$ $i < n$.

Recall def of CM. $x_1, \dots, x_n$ local parameters.

1) $x_i : F \to F$ is injective

2) F/x_1F is CM $\qquad H = (x_1 = 0)$.

$$0 \to F \xrightarrow{x_1} F \to F/x_1F \to 0$$

$$\underset{\substack{\| \\ 0}}{H^{i-1}_p(H, F/x_1F)} \to H^i_p(X, F) \hookrightarrow H^i_p(X, F) \qquad \text{if } i < n.$$

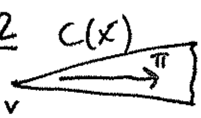
$$\Rightarrow H^i_p(X, F) = 0.$$

▶ **Exercises**:

HW1: If $p \in X$ smooth, local coord $x_1, x_2, \dots, x_n$. $\qquad H^n_p(X, \mathcal{O}_X) \simeq \langle \prod_j x_j^{-m_j} : m_j \geq 1 \rangle$

socle $\quad x_1^{-1} \cdots x_n^{-1}$.

HW2 $\quad C(X)$



$$C(X) = \operatorname{Spec} \sum_{m \geq 0} H^0(X, L^m). \qquad 0 \to H^0_v(C(X), \mathcal{O}) \to H^0(C(X), \mathcal{O}) \to H^0(C(X)^*, \mathcal{O})$$

$$\pi_* \mathcal{O}_{C^*(X)} = \sum_{m \in \mathbb{Z}} L^m.$$

For $i \geq 1$:
$$H^{i+1}_v(C(X), \mathcal{O}) \simeq H^i(C(X)^*, \mathcal{O}) = \sum_{m \in \mathbb{Z}} H^i(X, L^m).$$