Robert Lazarsfeld
Pretalk

▶ Amplitude (or bigness) of divisor D depends only on numerical class of D.

- **Nakai's Criterion**: D ample $\Longleftrightarrow$ $(D^k . V) > 0$ $\forall\, V$ dim k.

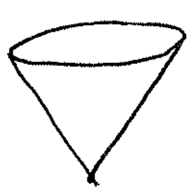
▶ Kleiman's Criterion:

- <u>Ask</u>: Can one limit oneself to $\dim V = 1$ in N.?
  Mumford: No.

- <u>Thm</u> (Kleiman) Assume (*) $(D \cdot C) \geq 0$ all curves C
  Then $D + \varepsilon H$ ample $\forall$ ample H $\varepsilon > 0$.
  i.e. D is a limit of ample divisors.

- <u>Def</u> D <u>nef</u> $(D . C) \geq 0$ all C.
  so "Nef is closure of ample".

▶ Cones: Let $N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. f. dim v.s./$\mathbb{R}$.



$N^1(X)_{\mathbb{R}} \supseteq \mathrm{Nef}(X) = \{\xi \mid (\xi . C) \geq 0 \text{ all } C\}$

- <u>Kleiman</u> $\mathrm{int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$.

$\mathrm{Psef}(X) = \mathrm{Pseudoeffcone}(X) = \left\{\begin{array}{l}\text{closed convex cone} \\ \text{spanned by all} \\ \text{effective divisors}\end{array}\right\}$.

- <u>Thm</u>: $\mathrm{int}(\mathrm{Psef}(X)) = \mathrm{Big}\, X$.

Robert Lazarsfeld.

▶ Positivity of Cycles on Abelian Varieties.

Problem: E nef vect. bundle. on X, $\dim X = 4$.
Is $c_2(E)$ represented. by (limit of) eff cycles?
(Allow $\mathbb{Q}$-coeffs).

$\boxed{X = \text{sm. proj var, } \dim n / \mathbb{C}}$

Joint w/ Debarre, Ein, Voisin.

▶ I. Review of Codim 1-picture.

Defs:

$$N^1(X) = N^1(X)_{\mathbb{R}} = \{\text{num equiv. classes}\} \qquad \text{f. dim v.s.}/\mathbb{R}$$

$$\cup\!\!|$$

$$\mathrm{Psef}(X) = \mathrm{Psef}^1(X) = \left\{\begin{matrix}\text{closed cone spanned}\\ \text{by eff divisors}\end{matrix}\right\}$$

$$\cup\!\!|$$

$$\mathrm{Nef}(X) = \{ \xi \mid (\xi \cdot C) \geq 0 \text{ all irred } C\}$$

Basic facts:

$$\mathrm{Big}(X) = \mathrm{int}(\mathrm{Psef}(X))$$

$$\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$$

• **Alternative View point**.

$$N_1(X) = \left\{\begin{matrix}\text{num. equiv classes of}\\ \mathbb{R}\ 1\text{-cycles}\end{matrix}\right\} = N^1(X)^\vee$$

$$\cup\!\!|$$

$$\mathrm{psef}_1(X) = \overline{NE}(X) = \left\{\begin{matrix}\text{closure of}\\ \text{cone of}\\ \text{curves}\end{matrix}\right\}$$

so $\mathrm{Nef}^1(X) = \mathrm{psef}_1(X)^\vee$ and $\mathrm{Nef}^1(X)^\vee = \mathrm{psef}_1(X)^\vee$

II Higher Codimension. Fix k. set

$N^k(X) = Z^k(X)_{\mathbb{R}} / \text{num eq.}$ f. dim v.s. $= N_k(X)^\vee$

<u>Def</u>: $\mathrm{Psef}^k(X) = \left\{ \begin{matrix} \text{closed convex cone spanned} \\ \text{by eff cycles} \end{matrix} \right\}$

$$\mathrm{Nef}^k(X) = \left\{ \alpha \in N^k(X) \;\middle|\; \begin{matrix} (\alpha \cdot z) \geq 0 \\ \text{all eff } Z \text{ dim } k \end{matrix} \right\} = \mathrm{psef}(X)^\vee$$

<u>Question</u>: What can one say about this?

<u>Grothendieck</u> (64' letter to Mumford).

Is the product of two nef classes again nef?

Esp, Is $\mathrm{Nef}^k(X) \subseteq \mathrm{Psef}^k(X)$?

<u>Corollary</u> (of thm to come later): Let E = elliptic curve with cx multiplication.

$X = E \times \cdots \times E$ (n times)

$\forall\ 2 \leq k \leq n-3$. X carries nef classes that are not pseudoeffective.

III. Positivity ~~for~~ on Abelian Varieties.

Say $B = V/\Lambda$ abelian variety. dimension n.

Numerical and homological equivalence agree on B.

$$N^k(B) \subseteq H^{k,k}(B) \cap H^{2k}(B, \mathbb{R}) = \{\text{real } (k,k) \text{ forms on } B\}$$

▶ Positivity of (k,k) forms on V.

consider η a real (k,k) form on V.

- **Def**: η is strongly positive if non-neg $\mathbb{R}$-lin comb of

$$i\phi_1 \wedge \bar{\phi}_1 \wedge i\phi_2 \wedge \bar{\phi}_2 \wedge \cdots \wedge i\phi_k \wedge \bar{\phi}_k \quad , \qquad \phi_\alpha \in V^*$$

- **Def** η is weakly positive if:
  $\eta|_{W^k} \geq 0$ all cx subspace $W^k \subseteq V$

- Give closed convex cones (*) $\quad \mathrm{Strong}^k(V) \subseteq \mathrm{Weak}^k(V)$

  * If $k = 1, n-1$ equality
  * If $2 \leq k \leq n-2$ strict inclusion in (*).

Back to $B = V/\Lambda$: $\quad N^k(B) \subseteq \left\{ \begin{matrix} \text{real } (k,k) \\ \text{forms on } V \end{matrix} \right\}$

Define: $\quad \mathrm{Strong}^k(B) = N^k(B) \cap \mathrm{Strong}^k(V).$

$\qquad\qquad \mathrm{Weak}^k(B) = \text{Similar}.$

- **Lemma**: Have

$$\mathrm{Psef}^k(B) \subseteq \mathrm{Strong}^k(B) \subseteq \mathrm{Weak}^k(B) \subseteq \mathrm{Nef}^k(B)$$

If $k = 1, n-1$ all coincide.

NB: If B is "very general" then $\dim N^k(B) = 1$.

- **Thm**: Let E ell curve with complex multiplication.

$$B = E \times \cdots \times E \qquad (n \text{ times})$$
$$= V/\Lambda.$$

Then,
$$\mathrm{Psef}^k(B) = \mathrm{Strong}^k(B) = \mathrm{Strong}^k(V)$$
$$\mathrm{Nef}^k(B) = \mathrm{Weak}^k(B) = \mathrm{Weak}^k(V).$$

So if $2 \leq k \leq n-2$, then
$$\mathrm{Psef}^k(B) \subsetneq \mathrm{Nef}^k(B)$$

pf/ $\mathrm{Psef}^1(B) = \mathrm{Strong}^1(V)$.

- **Thm** A = Very general ppal. polarized abel. surface. $X = A \times A$.

Then: $\mathrm{Psef}^2(B) = \mathrm{Strong}^2(B) \subsetneq \mathrm{Weak}^2(B) \subsetneq \mathrm{Nef}^2(B)$.

(1)
Effectivity of Char. Classes

Define cones $\Sigma_{k,n}$, $\Pi_{k,n} \subseteq \mathbb{R}[c_1, \ldots, c_n]_{\text{wtd deg } k}$.

$$\Sigma_{k,n} = \left\{ P \;\middle|\; \begin{array}{c} P(c_1(E), \ldots, P(c_n(E)) \in \mathrm{Psef}^k(X) \\ \forall \text{ nef } E \text{ on all } X^n \end{array} \right\}$$

$$\Pi_{k,n} = \left\{ P \;\middle|\; P(c(E)) \in \mathrm{Nef}^k(X) \right\}$$

- Fulton - Lazarsfeld (80's)

  $\Pi_{k,n}$ = generated by Schur poly S_λ.

Rmk $\Sigma_{k,n} \subseteq \Pi_{k,n}$

- Problem: What is $\Sigma_{k,n}$?

  Is it $\Sigma_{k,n} \subsetneq \Pi_{k,n}$?

(2) When is a cycle big?

Def (Peternell) $\mathrm{Big}^k(X) = \mathrm{int}\,(\mathrm{Psef}^k(X))$

Want to characterize $\mathrm{Big}^k(X)$.

Peternell If $Y \subseteq X$ w $N_{Y/X}$ ample. Is $[Y]$ big?

Voisin Found counterexample for $Y^2 \subseteq X^4$.

Asymtotic approach?

- Conjecture: class $\beta \in N^k(X)_{\mathbb{Z}}$
  big $\Leftrightarrow$ following holds:

$\exists\, c > 0$, arb large $m \in \mathbb{N}$, and all eff cycle $Z_m \sim m\beta$ passing through $\geq c \cdot m^{n/k}$ very general points.

Big $\Rightarrow$ Cond.

- OK: if $k = 1, n-1$
- OK: if $k = 2$, $\operatorname{Pic} X = \mathbb{Z}$.