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Universal Formulas for Counting Curves on Surfaces

S: smooth proj surface / $\mathbb{C}$

L: line bundle on S

Q: How many reduced curves in $|L|$ have r simple nodes and no other singularities.

notation: r-nodal curves.

previous results:

① $S = \mathbb{P}^2$ (Ran, Manin, Kontsevich, Harris, Pandharipande, Choi)

- $\mathcal{O}(d)$ rat. nodal curves: Kont-Manin 1994.
- $\mathcal{O}(d)$ arb. genus: Caporaso - Harris.
- (2009) Fomin-Mikhalkin: # r-nodal curves in $\mathcal{O}(d)$ is a poly in d $\forall d \geq 2r$.

② S rat. ruled sf. (Vakil)

③ $S = K3$. L a primitive l.b. ($\mathrm{Pic}\, S = \mathbb{Z}L$)

① rational Yau-Saslow $\displaystyle\sum_{r=0}^{\infty} \binom{\#\ r\text{-nodal}}{\text{ratl curves}} q^{r-1} = \frac{1}{\Delta}$ $\quad \Delta = q\prod_{k>0}(1-q^k)^{24}$

② arbitrary Genus (Bryan - Leung) $\displaystyle\sum_{r=0}^{\infty} \binom{\#\ r\text{-nodal}}{\text{curves in } |L|} (DG_2)^r = \frac{DG_2/q}{\Delta D^2 G_2/q}$

G_2 second Eisenstein fcn $D = q \cdot \frac{d}{dq}$

$G_2 \,.\, DG_2 \,.\, D^2 G_2$ quasi-modular forms.

- On all alg. surfaces: $r \leq 3$ standard intersection theory
- Vainsencher $r \leq 6$ L "suff" ample.
- Kleiman-Piene $r \leq 8$

$r=1$ $T_1 = 3L^2 + 2LK + c_2(S)$

$r=2$ $T_2 = \dfrac{T_1(-7+T_1) - 6c_1(S)^2 - 25LK - 21L^2}{2}$

$T_3 : \cdots$ $T_4 : \cdots$ $T_5 : \cdots$ $T_6 : \ldots$ (half page) $\cdots$

▶ Thm (Gottsche's Conjecture)

For all smooth proj surfaces$/\mathbb{C}$, L is $(5r-1)$-very ample $\forall r \geq 0$. there exist a univ. poly T_r. s.t.

$$T_r(L^2, LK, c_1(S)^2, c_2(S)) = \#\ r\text{-nodal curves in } |L|.$$

L is $(5r-1)$-very ample means $H^0(L) \to L|_\xi$ $\xi \in S^{[5r]}$.

▷Rmk: ① A symplectic Pf Liu (2000)
② T_r univ indep of S.L.
③ # of nodal curves only dep. on $L^2, LK, c_1(S)^2, c_2(S)$
④ T_r can be computed by cases on $\mathbb{P}^2$ and K3.

▶ **Theorem** [B]

$$\sum_{r=0}^{k} \mathrm{Tr}\left(L^2, Lk, c_1(S)^2, c_2(S)\right) \cdot x^r = A_1^{L^2} A_2^{LK} A_3^{c_1(S)^2} \cdot A_4^{c_2(S)}$$

$A_i \in \mathbb{Q}[[x]]^*$ unknown.

▶ **Theorem** [C]

$$\sum_{r=0}^{k} \mathrm{Tr}\left(L^2, Lk, c_1(S)^2, c_2(S)\right) (DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK}}{(\Delta \quad D^2G_2/q^2)^{\chi \cdot 1057/2}}$$

$B_1, B_2 \in \mathbb{Q}[[q]]$

Rmk Thm [C] can be proved ~~by~~ from Thm [B] + Bryan-Leung's formula.

▶ **Approach**: · Algebraic Cobordism
· Degeneration formula.

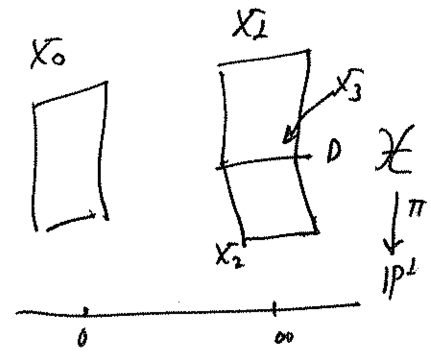
$$\begin{tikzcd}
 & \mathbb{Q}[[x]]^* & \\
W_{2,1} \arrow[ur, "\text{deg. formula}"] \arrow[rr, "\sim", "\text{alg cob}"'] & & \mathbb{Q}^4 \arrow[ul, "\text{Thm}"']
\end{tikzcd}$$

alg cob.

④

▶ Algebraic Cobordism.

Morel and Levine, Levine and Pandharipande.



Def X_i = proj sm. schemes.

Double Point formula. $[X_0] = [X_1] + [X_2] - [X_3]$ if

$\exists \ \mathcal{X} \longrightarrow \mathbb{P}^1$ s.t.

① $\mathcal{X}$ smooth π smooth except $\pi^{-1}(\infty)$

② X_0 smooth fiber over 0.

③ $\pi^{-1}(\infty) = X_1 \cup_D X_2$ intersect transversally along a smooth divisor D.

④ $X_3 = \mathbb{P}(\mathcal{O}_D \oplus N_{X_1/D}) \simeq \mathbb{P}(N_{X_2/D} \oplus \mathcal{O}_D)$.

(Rmk: You can write a scheme as a sum of simpler schemes).

▶ Thm $W_* = \sum \mathbb{Q}[X_i] \big/ \text{d.p. formula}.$

"Use degeneration to study the number of nodal curves."

Levine-P.

$$W_* = \bigoplus_{\substack{\lambda \\ = (\lambda_1, \dots, \lambda_r)}} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \mathbb{P}^{\lambda_2} \times \dots \times \mathbb{P}^{\lambda_r}]$$

If S smooth proj sf.

$$\Rightarrow \quad [S] = * [\mathbb{P}^2] + * [\mathbb{P}^1 \times \mathbb{P}^1]$$

- Def: $[X_i, L_i]$ pairs of sm proj sf and L_i l.b. on X_i.

- Extended double point relation.

$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3] \quad \underline{\underline{\text{if}}}$$

① $[X_0] = [X_1] + [X_2] - [X_3]$ is a d.p. relation.

② There exist a l.b. $\mathcal{L}$ on $\mathcal{X}$ s.t. $\mathcal{L}|_{X_i} = L_i \ \forall i = 0,1,2.$

③ L_3 is the pull back of $\mathcal{L}|_D$ to X_3 via $X_3 \to D$.

▶ Thm: $W_{2,1} = \bigoplus \mathbb{Q}[X_i, L_i] \Big/ \substack{\text{extended} \\ \text{d.p. relation}}$

$$W_{2,1} \xrightarrow[(L^2,\, LK,\, c_1(S)^2,\, c_2(S))]{\sim} \mathbb{Q}^4$$

Generators.

$$\{ [\mathbb{P}^2, 0],\ [\mathbb{P}^2, \mathcal{O}(1)],\ [\mathbb{P}^1 \times \mathbb{P}^1, 0],\ [\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)] \}$$

- **Difficulty:** ampleness is not preserved by degeneration.

Sol:

$$\begin{tikzcd} & Z \arrow[r, "q"] \arrow[d, "p"] & S^{[n]} \\ L \arrow[r, "\text{def}"] & S & \end{tikzcd}$$

Z : univ closed subsch.

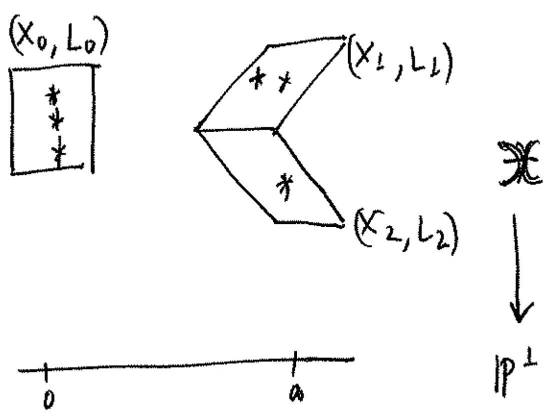
$q_* p^* L = L^{[n]}$ v.b. on $S^{[n]}$

$$W^{3r} = \text{the closure of } \left\{ \coprod_{i=1}^{r} \operatorname{Spec} \mathcal{O}_{x_i,S} / m^2_{x_i,S} \;\middle|\; \begin{array}{l} x_i \text{ distinct} \\ \text{pts on } S \end{array} \right\} \subseteq S^{[3r]}$$

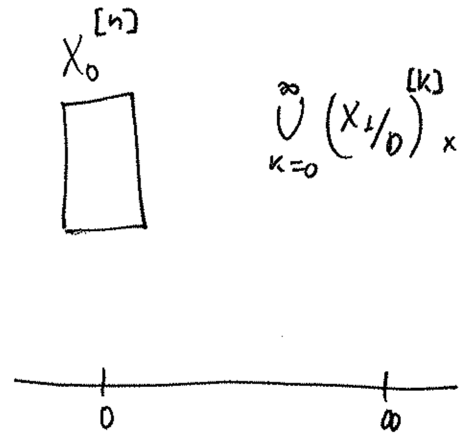
$$\operatorname{div}(S, L) := \int_{W^{3r}} c_{2r}\left(L^{[3r]}\right)$$

Fact: $\operatorname{div}(S, L) = \#$ of r-nodal curves.

If L is $(3r-1)$ very ample.



Li - Wu



$$\bigcup_{k=0}^{\infty} \left(X_1/D\right)^{[k]} \times \left(X_2/D\right)^{[n-k]}$$

a sample pt. of $\left(X_1/D\right)^{[3]}$ 

$$\phi(S,L) = \sum \operatorname{div}(S,L)\, x^r$$

Prop $\phi(X_0, L_0) = \phi(X_1/D, L_1)\,\phi(X_2/D, L_2)$

Thm $$\phi(X_0, L_0) = \frac{\phi(X_1, L_1)\cdot\phi(X_2, L_2)}{\phi(X_3, L_3)}$$

if $[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$

is a extended d.p. relation

- Cor: φ is a homomorphism.

from $W_{2,1} \longrightarrow Q[\![X]\!]^*$

$$\begin{tikzcd} & \mathbb{Q}[\![X]\!]^* & A_i \\ W_{2,1} \arrow[ur, "\varphi\ \text{homo}"] \arrow[rr, "\sim"] & & \mathbb{Q}^4 \arrow[ul] & e_i \arrow[ul, mapsto] \end{tikzcd}$$

$[L^2, LK, c_1(S)^2, c_2(S)]$.

$$\sum \operatorname{div}(S,L)\, x^r = \phi(S,L) = A_1^{L^2} A_2^{LK} \cdot A_3^{c_1(S)^2} \cdot A_4^{c_2(S)}$$

$\operatorname{div}(S,L)$ is univ $=\!=$ # r-nodal cone very ample if $L-(5r-1)$ very ample

$\Longrightarrow$ Proves Thm [A], [B].