PROBLEM SESSION

AGNES, SPRING 2011

For a full video of the problem presentations, visit `http://techtv.mit.edu/collections/mathagnes`.

János Kollár

Problem 1: Intersection Homology and the Picard Group

Given an algebraic variety over $\mathbb{C}$ or complex analytic space, there's a standard exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$. Analytically, this gives rise to an exact sequence

$$H^1(X, \mathcal{O}_X^{an}) \longrightarrow \mathrm{Pic}^{an}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

Question. Can we put intersection cohomology $IH^2(X, \mathbb{Z})$ in the last term and still get an exact sequence?

This is interesting if X is singular, where intersection cohomology behaves much better than ordinary cohomology. Arapura indicated that this should be true if X is projective.

Problem 2: A Lefschetz-type Theorem

Given a variety X and point $0 \in X$, suppose a Cartier divisor X_0 passing through 0 is given. Suppose there is a map $X \to \Delta$ with fiber X_0 over $0 \in \Delta$, so that X is the total space of a deformation of X_0.

Question. When is the map $\mathrm{Pic}(X \setminus 0) \to \mathrm{Pic}(X_0 \setminus 0)$ injective?

There are some results about this for a general X_0, but we're interested in working with a specific X_0. In SGA 2, it is shown that the map is injective if X_0 is S_3.

Question. Assume that $\dim X_0 \geq 3$ and X_0 is (normal, S_2). Does this imply that that $\mathrm{Pic}(X \setminus 0) \to \mathrm{Pic}(X_0 \setminus 0)$ is an injection?

There are examples where a divisor is Cartier on every fiber but is not Cartier. The standard examples consist of families of surfaces, and this problem asserts that the problem doesn't arise in higher dimensions.

Problem 3: Terminal Singularities in Dimension ≥ 4

In dimension 3, any terminal singularity can be written as a quotient $X = \tilde{X}/G$, where $\tilde{X} \subset \mathbb{C}^4$ with multiplicity 2 and G is a cyclic group.

Question. Is it possible that every terminal X of dimension n can be written as $\tilde{X}/G$, $\tilde{X} \subset \mathbb{C}^{\alpha(n)}$, $\mathrm{mult}\, \tilde{X} \leq \beta(n)$.

It would be surprising if the answer is "yes", but no counterexample is known.

Problem 4: Restrictions of Stable Bundles

Theorem (Mehta-Ramanathan)**.** Suppose S is a smooth projective surface, $|H|$ a very ample linear system, and E a stable vector bundle on S. Then for a general curve $C \subset |mH|$, $E|_C$ is also stable.

One may also ask how large this m must be. There are some effective results; for example, Flenner showed that $m \sim (\operatorname{rk} E)^4 \cdot (H^2)$.

Question. Why not take $m = 4$? Is there a simple counterexample?

Bjorn Poonen

These problems are intended to be very hard – the goal is to make them so hard that they can be proved undecidable.

Problem 1: The Special Set

Let k be a number field and X be a variety over k. It is natural to ask what properties of X would force the set of rational points to be infinite. Obvious answers are the existence of a map $\mathbb{P}^1 \dashrightarrow X$ defined over k, or an elliptic curve with infinitely many rational points mapping to X.

Lang defined the *special set* S of X to be the Zariski closure of the union of images $f(A)$ where $f\colon A \dashrightarrow X$ ranges over rational maps from an abelian variety to X, defined over the algebraic closure $\bar{k}$. He conjectured that $(X \setminus S)(k)$ is always finite. This is a vast generalization of the Mordell conjecture. But it might be useless in practical terms:

Question (Mazur)**.** Can one decide whether $S = \emptyset$?

Question. Can one decide whether X contains a rational curve?

One approach would be to try to reduce to some other problem known to be undecidable. For example, it is known to be undecidable whether a polynomial equation in integers has an integer solution.

Some problems like those in the questions above are probably decidable. For example,

Question. Can one decide whether X is rationally connected?

A conjecture depending on the MMP states that X is rationally connected if and only if $H^0(X, (\Omega^1_X)^{\otimes n}) = 0$ for all n. Just search both for a family of rational curves connecting points on X, and an n such that this sheaf has sections. One of the searches must eventually stop, and this tells whether X is rationally connected.

Problem 2: Automorphisms

Let X be a smooth projective variety over $\mathbb{C}$. There's the usual exact sequence

$$0 \longrightarrow \operatorname{Pic}^0(X) \longrightarrow \operatorname{Pic} X \longrightarrow \operatorname{NS}(X) \longrightarrow 0.$$

An automorphism of X acts on each of these groups, and so we could try to understand automorphisms of X by studying these actions.

Let $\mathcal{A}_X = \operatorname{im}(\operatorname{Aut} X \to \operatorname{Aut}(\operatorname{NS} X))$. This is a discrete image of the automorphism group. For example, if $\operatorname{NS}(X)$ is free then the group on the right hand side is $\operatorname{GL}_n(\mathbb{Z})$ and $\mathcal{A}_X$ is some subgroup thereof.

Question. Is $\mathcal{A}_X$ always finitely generated? Finitely presented?

Question. Given a finitely presented group G, can one construct X with $\mathcal{A}_X \cong G$?

Question. Can one decide whether X has a nontrivial automorphism?

At least one related question is undecidable.

Theorem (Poonen)**.** Let $Z \subset X$ be a closed subvariety, and $x \in X$ a point. It is undecidable whether there exists $\alpha \in \operatorname{Aut} X$ with $\alpha(x) \in Z$.

Problem 3: Sections of Rational Maps

Question. Can one decide whether a rational map $X \dashrightarrow \mathbb{P}^1$ has a rational section? Equivalently, is it decidable whether a polynomial over $\bar{\mathbb{Q}}(t)$ has a rational solution?

It is known that this is undecidable if $\mathbb{P}^1$ is replaced with $\mathbb{P}^2$. This was proved by Kim and Roush, and generalized by Eisentraeger to the case where $\mathbb{P}^2$ is replaced by any fixed variety of dimension at least two.

Problem 4: Isomorphisms

Burt Totaro suggested to me that perhaps the following problem is undecidable.

Question. Given two varieties X and Y, perhaps smooth and projective over $\bar{\mathbb{Q}}$, can one decide whether or not they are isomorphic?

Question. Given two finitely presented commutative algebras over $\bar{\mathbb{Q}}$ (or $\mathbb{Q}$, or $\mathbb{Z}$), can one decide whether they are isomorphic?

This is not even known over $\mathbb{Z}$. It's known to be undecidable without the assumption of commutativity; if G is a finitely presented group, checking whether $\mathbb{Z}G \simeq \mathbb{Z}$ amounts to deciding whether G is trivial, which is a well-known undecidable problem.

Jason Starr

Problem 1: Fixed Points of Automorphisms

Corollary (of the Atiyah-Bott theorem)**.** Let X be a smooth projective variety over $\mathbb{C}$, with an action of $\mathbb{Z}/\ell\mathbb{Z}$ on X by an algebraic isomorphism. Then if $h^q(X, \mathcal{O}_X) = 0$ for all $q > 0$, the action has a fixed point.

In terms of the Hodge diamond, this states that the outermost ring has all 0s, then there must be a fixed point.

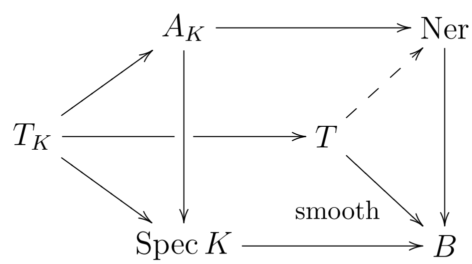
What if the action is by $\Gamma = \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$? The answer can't be quite so simple: the Γ-linearized line bundles $\operatorname{Pic}^\Gamma(X)$ map to the Γ-invariant line bundles $\operatorname{Pic}(X)^\Gamma$. If Γ has a fixed point, then the map $\phi : \operatorname{Pic}^\Gamma(X) \to \operatorname{Pic}(X)^\Gamma$ is surjective.

Question. Assume this "elementary obstruction" is 0. What condition can we impose that will assure the existence of a fixed point?

If $X = \mathbb{P}^n$ or $X = G/P$ is a projective homogeneous variety, this is enough. If $X_d \subset \mathbb{P}^n$ is a hypersurface, the condition holds when $d \leq n$, and there is a fixed point when $d^2 \leq n$, but there are counterexamples with $d^2 = n + 1$.

Problem 2: Néron models

Suppose B is a one-dimensional regular scheme, with generic point $\mathrm{Spec}(K) \to B$. The Néron model satisfies the following universal property: given an abelian variety over K, there exists the Néron model over B such that given any scheme T smooth over B with a map to A_K over the generic point, the map extends to the Neron model.



Nowhere in the statement is it important for this property that A is an abelian variety (of course in the construction it is!)

Question. Are there Néron models for other types of varieties?

If $X_K \to \mathrm{Alb}(X_K)$ is an embedding, then the Zariski closure in $\mathrm{Ner}(A_K)$ may be taken as $\mathrm{Ner}(X_K)$; so the answer is "yes" for varieties which embed in the Albanese.

Question. Are there any simply connected examples? (In this case the Albanese is trivial.)

Problem 3: Rational Curves on Moishezon Manifolds

One consequence of bend and break is that if X is a smooth projective variety over $\mathbb{C}$, and $C \subset X$ is a curve such that $K_X \cdot C < 0$, then for every point $p \in C$ there exists a nonconstant morphism $f : \mathbb{P}^1 \longrightarrow X$ whose image contains p.

Question. Is the same true if X is a smooth proper algebraic space (i.e. compact Moishezon manifold)?