Gromov-Witten invariants and puzzles

Anders Buch

Collaborators: A. Kresch, L. Mihalcea, K. Purbhoo, H. Tamvakis

Grassmann variety: $X = \mathrm{Gr}(m, n) = \{V \subset \mathbb{C}^n \mid \dim(V) = m\}$

$\dim_{\mathbb{C}}(X) = mk$, where $k = n - m$.

Def: A Schubert variety in X is an orbit closure for the lower triangular matrices in $\mathrm{GL}(n)$.

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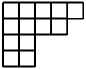
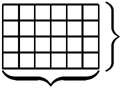
1-1 Correspondence: Partitions $\leftrightarrow$ Schubert varieties

Partition: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ =  $\subset$  m

k

Corresponding Schubert variety:

$X_\lambda = \{V \in X \mid \dim(V \cap \mathbb{C}^{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}$.

$\mathrm{codim}(X_\lambda; X) = |\lambda| = \sum \lambda_i = \#$ boxes in Young diagram.

Schubert calculus

$$H^*(X;\mathbb{Z}) = \bigoplus_{\lambda} \mathbb{Z}\,[X_\lambda] \quad ; \quad [X_\lambda]\cdot[X_\mu] = \sum_{\nu} c^{\nu}_{\lambda,\mu}[X_\nu]$$

$c^{\nu}_{\lambda,\mu}$ = Littlewood-Richardson coefficient

Geometric formula: $c^{\nu}_{\lambda,\mu} = \#(g_1.X_\lambda \cap g_2.X_\mu \cap g_3.X_{\nu^\vee})$

where $g_1, g_2, g_3 \in \mathrm{GL}(n)$ generic matrices

and $\nu^\vee = (k-\nu_m, k-\nu_{m-1}, \dots, k-\nu_1)$ Poincare dual partition.

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Example:

$X = \mathrm{Gr}(2,4) = \{$ lines in $\mathbb{P}^3$ $\}$

$X_{(1)} = X_\square = \{$ lines in $\mathbb{P}^3$ meeting the line $\mathbb{C}^2$ $\}$

Lines in $\mathbb{P}^3$ meeting 4 fixed lines L_1, L_2, L_3, L_4 ; $L_i = g_i.\mathbb{C}^2$, $g_i \in \mathrm{GL}(4)$:

$$g_1.X_\square \cap g_2.X_\square \cap g_3.X_\square \cap g_4.X_\square$$

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Compute number of lines in $H^*(X;\mathbb{Z})$:

$$[g_1.X_\square \cap g_2.X_\square \cap g_3.X_\square \cap g_4.X_\square] = [X_\square]^4 = 2\,[X_{\boxplus}] = 2\,[\text{point}]$$

Gromov-Witten invariants

Def: A (rational) **curve** $C \subset X$ is any image of a polynomial map $\mathbb{P}^1 \to X$.

Degree: $\deg(C) = \#(C \cap X_{\square})$

Same as degree in Plücker embedding $C \subset X \subset \mathbb{P}^N$, $N = \binom{n}{m} - 1$.

Note: Point = curve of degree zero!

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Def: Given λ, μ, ν with $|\lambda| + |\mu| + |\nu| = \dim(X) + nd$, define
$I_d(X_\lambda, X_\mu, X_\nu) = \#$ curves $C \subset X$ of degree d
meeting $g_1.X_\lambda$, $g_2.X_\mu$, and $g_3.X_\nu$.

Example: $c^{\nu}_{\lambda,\mu} \; = \; \#(g_1.X_\lambda \cap g_2.X_\mu \cap g_3.X_{\nu^\vee}) \; = \; I_0(X_\lambda, X_\mu, X_{\nu^\vee})$

Small quantum cohomology ring:

$QH(X)$ is a $\mathbb{Z}[q]$-algebra deforming $H^*(X;\mathbb{Z})$.

$$QH(X) = H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q] = \bigoplus_{\lambda} \mathbb{Z}[q] \cdot [X_\lambda]$$ as a $\mathbb{Z}[q]$-**module**

$$[X_\lambda] \star [X_\mu] = \sum_{\nu, d \geq 0} I_d(X_\lambda, X_\mu, X_{\nu^\vee})\, q^d\, [X_\nu]$$

Thm: (Ruan & Tian, Kontsevich & Manin)

This product is associative !!!!!

Note: $QH(X) \,/\, (q = 0) \;=\; H^*(X;\mathbb{Z})$

Structure theorems for $QH(X)$

Set $\sigma_p = [X_{(p)}]$ for $1 \le p \le k$, $\sigma_0 = 1$, and $\sigma_p = 0$ for $p < 0$ or $p > k$.

For $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell)$, define

$$\Delta_\lambda = \det\left(\sigma_{\lambda_i+j-i}\right)_{\ell\times\ell} = \det\begin{bmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \cdots & \sigma_{\lambda_1+\ell-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \cdots & \sigma_{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\lambda_\ell-\ell+1} & \sigma_{\lambda_\ell-\ell+2} & \cdots & \sigma_{\lambda_\ell} \end{bmatrix}$$

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Presentation: (Witten , Siebert & Tian)

$$QH(X) = \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_k, q]/(\Delta_{(1^{m+1})}, \Delta_{(1^{m+2})}, \ldots, \Delta_{(1^{n-1})}, \Delta_{(1^n)} + (-1)^k q)$$

Quantum Giambelli formula: (Bertram): $[X_\lambda] = \Delta_\lambda$ in $QH(X)$.

Application: Compute $I_d(X_\lambda, X_\mu, X_\nu)$:

Solve equation $\Delta_\lambda \star \Delta_\mu = \sum_{\nu, d\geq 0} I_d(X_\lambda, X_\mu, X_\nu)\, q^d\, \Delta_{\nu^\vee}$ in presentation.

Kernel and Span

Let $C \subset X = \mathrm{Gr}(m,n)$ be a curve.

Def: (B) $\mathrm{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n$ and $\mathrm{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$

Obs: $\dim \mathrm{Ker}(C) \geq m - \deg(C)$ and $\dim \mathrm{Span}(C) \leq m + \deg(C)$

Application: (B) Much simpler proofs of structure theorems for $QH(X)$.

Quantum = classical theorem

Def: $Y_d = \mathrm{Fl}(m-d, m+d; n)$
$$= \{(A,B) \mid A \subset B \subset \mathbb{C}^n \;,\; \dim(A) = m-d \;,\; \dim(B) = m+d\}$$

Given Schubert variety $X_\lambda \subset X$, define

$$\widetilde{X}_\lambda = \{(A,B) \in Y_d \mid \exists V \in X_\lambda : A \subset V \subset B\}$$

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Thm: (B & Kresch & Tamvakis) Explicit bijection:

$$\left\{ \begin{array}{c} \text{curves } C \subset X \\ \text{of degree } d \\ \text{meeting } g_1.X_\lambda\ , \\ g_2.X_\mu\ ,\ g_3.X_\nu \end{array} \right\} \longleftrightarrow \quad g_1.\widetilde{X}_\lambda \cap g_2.\widetilde{X}_\mu \cap g_3.\widetilde{X}_\nu \quad \subset Y_d$$

$$C \quad \mapsto \quad (\mathrm{Ker}(C), \mathrm{Span}(C))$$

Cor: $I_d(X_\lambda, X_\mu, X_\nu) = \displaystyle\int_{Y_d} [\widetilde{X}_\lambda] \cdot [\widetilde{X}_\mu] \cdot [\widetilde{X}_\nu]$

Two-step flag variety: Let $0 < a < b < n$.

$Y = \mathrm{Fl}(a, b; n) = \{(A, B) \mid A \subset B \subset \mathbb{C}^n \; ; \; \dim(A) = a \; ; \; \dim(B) = b\}$

Def: A **012-string** for Y is a permutation of $\mathbf{0} := 0^a \, 1^{b-a} \, 2^{n-b}$.

E.g. $u = 10212$ is a 012-string for $\mathrm{Fl}(1, 3; 5)$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $u = (u_1, u_2, \ldots, u_n)$ 012-string.

Set $A_u = \mathrm{Span}\{e_i : u_i = 0\}$ and $B_u = \mathrm{Span}\{e_i : u_i \leq 1\}$.

$\mathbf{B} \subset \mathrm{GL}(\mathbb{C}^n)$ lower triangular matrices.

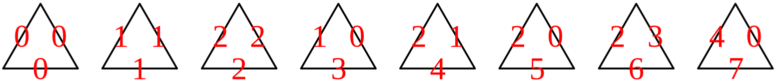
Schubert variety: $Y_u = \overline{\mathbf{B}.(A_u, B_u)}$

$\mathrm{codim}(Y_u; Y) = \ell(u) = \#\{i < j \mid u_i > u_j\}$

Schubert structure constants:

$H^*(Y; \mathbb{Z}) = \bigoplus_u \mathbb{Z}[Y_u]$; $[Y_u] \cdot [Y_v] = \sum_w c_{u,v}^w [Y_w]$

Def: (Knutson) A **puzzle piece** is a (small) triangle from the following list:



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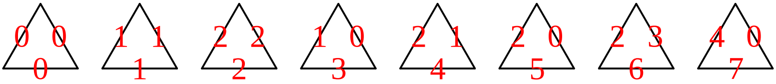


May be rotated, but not reflected.

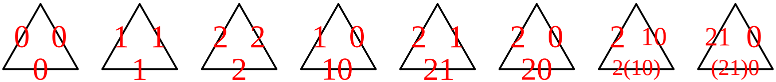
Interpretation of labels as **decreasing trees of integers**:

Simple labels:

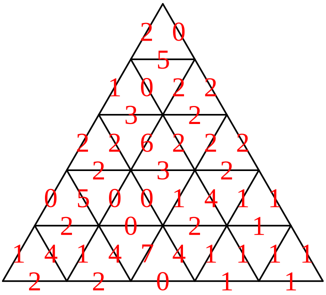
0 1 2

Composed labels:

3 = 10 4 = 21, 5 = 20, 6 = 2(10), and 7 = (21)0



Def: (Knutson) A **puzzle** is a triangle made from puzzle pieces with matching side labels.



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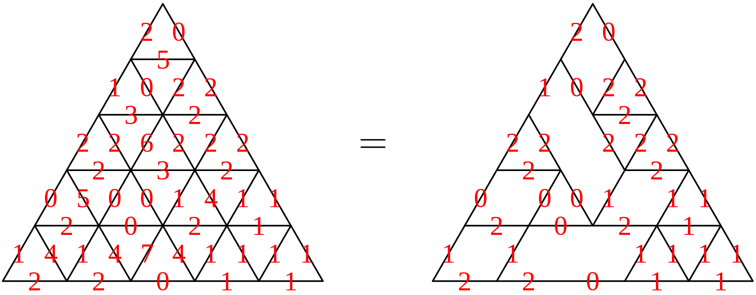
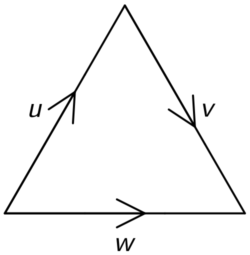


Note: The composed labels are uniquely determined by simple labels.

Conjecture (Knutson) /
Theorem: (Buch & Kresch & Purbhoo)

$c_{u,v}^{w} = \#$ puzzles with border labels u, v, w :



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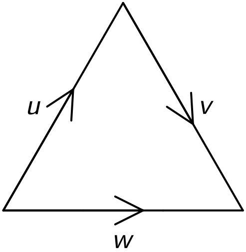
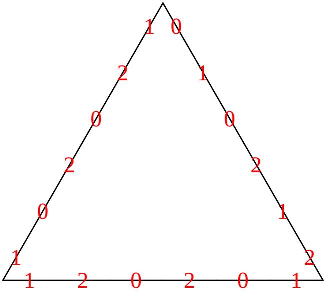


Example: $u = 102021$, $v = 010212$, $w = 120201$: $c_{u,v}^{w} =$?



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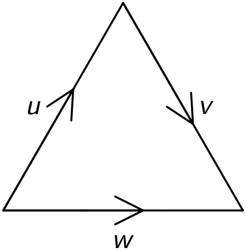
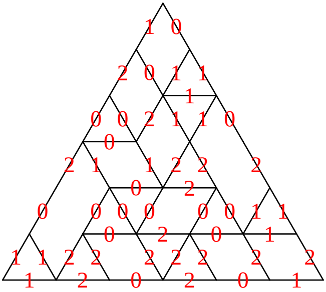
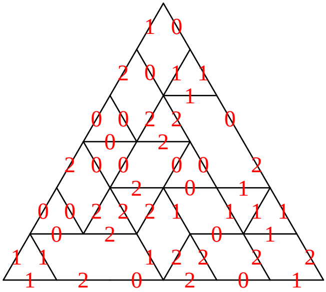


Example: $u = 102021$, $v = 010212$, $w = 120201$: $c_{u,v}^{w} = 2$





Quantum Littlewood-Richardson rule

$X_\lambda \subset X = \mathrm{Gr}(m, n)$ Schubert variety.

$Y = Y_d = \mathrm{Fl}(m - d, m + d; n)$.

$\widetilde{X}_\lambda = Y_{u(\lambda,d)} \subset Y_d$

for some 012-string $u(\lambda, d)$.

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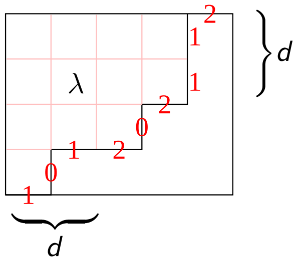

$u(\lambda, d) = (1, 0, 1, 2, 0, 2, 1, 1, 2)$

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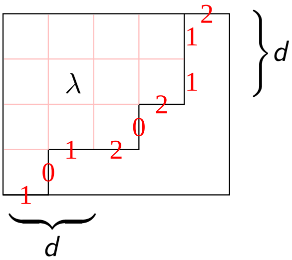


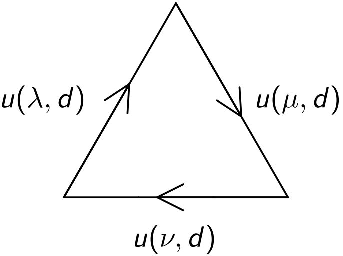
$u(\lambda, d) = (1,0,1,2,0,2,1,1,2)$

Corollary:

$I_d(X_\lambda, X_\mu, X_\nu)$

$$= \int_{Y_d} [Y_{u(\lambda,d)}] \cdot [Y_{u(\mu,d)}] \cdot [Y_{u(\nu,d)}]$$

$=$ # puzzles with border labels $u(\lambda,d)$, $u(\mu,d)$, $u(\nu,d)$.



1999: Knutson circulated puzzle conjecture for all partial flag varieties $SL(n)/P = Fl(a_1, a_2, \ldots, a_m; n)$.

Shortly after: Knutson found counter example for $Fl(1, 2, 3, 4; 5)$.

2001: Knutson, Tao, Woodward proved puzzle rule for $Gr(m, n)$.

2001: Knutson and Tao proved generalization for $H^*_T(Gr(m, n))$.

2002: Buch, Kresch, Tamvakis: All (3-point, genus zero) Gromov-Witten invariants of degree d on $Gr(m, n)$ are equal to Schubert structure constants $c^w_{u,v}$ of $Fl(m-d, m+d; n)$.

Suggested that conjecture is true for two-step flag varieties.

Verified conjecture for all $Fl(a, b; n)$ with $n \leq 16$.

2007: Coskun proved different LR rule for $Fl(a, b; n)$ using Mondrian tableaux.
Based on degenerating intersection of Schubert varieties.

2010: Knutson and Purbhoo proved that special case of Knutson's original conjecture for $SL(n)/P$ computes Belkale-Kumar coefficients.

Equivariant cohomology

$T = (\mathbb{C}^*)^n \subset \mathrm{GL}(n)$ acts on $X = \mathrm{Gr}(m, n)$.

$H_T^*(X; \mathbb{Z})$ is a $\mathbb{Z}[y_1, \dots, y_n]$-algebra deforming $H^*(X; \mathbb{Z})$.

$$H_T^*(X; \mathbb{Z}) = H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[y_1, \dots, y_n] = \bigoplus_{\lambda} \mathbb{Z}[y_1, \dots, y_n] \cdot [X_\lambda]_T$$

as a $\mathbb{Z}[y_1, \dots, y_n]$-module.

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Thm: (Knutson & Tao) Puzzle formula for polynomials $c_{\lambda,\mu}^\nu$.

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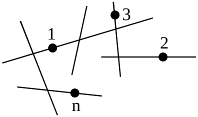
Let $\rho : X \to \{\text{point}\}$ be the map to a point.

Def: For $\alpha \in H_T^*(X;\mathbb{Z})$ we set

$$\int_X^T \alpha \;=\; \rho_*(\alpha) \;\in H_T^*(\text{point}) = \mathbb{Z}[y_1, \ldots, y_n].$$

Kontsevich moduli space

Let $d \in H_2(X;\mathbb{Z})$.

$C =$ 

$\overline{\mathcal{M}}_{0,n}(X,d) = \{\text{stable } f : C \to X \mid f_*[C] = d\,[\text{line}]\}$

Evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$; $\mathrm{ev}_i(f) = f(i\text{-th marked point})$

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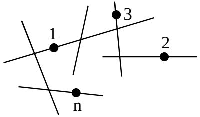
$C =$ 

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Gromov-Witten invariant: Given $\alpha_1, \alpha_2, \ldots, \alpha_n \in H^*(X)$ define

$$I_d(\alpha_1, \alpha_2, \ldots, \alpha_n) = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdots \mathrm{ev}_n^*(\alpha_n)$$

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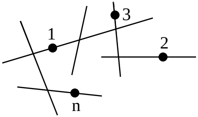
$C =$ 

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Evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$; $\mathrm{ev}_i(f) = f(i\text{-th marked point})$

Gromov-Witten invariant: Given $\alpha_1, \alpha_2, \ldots, \alpha_n \in H^*(X)$ define

$$I_d(\alpha_1, \alpha_2, \ldots, \alpha_n) = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdots \mathrm{ev}_n^*(\alpha_n)$$

Key application: Used to prove associativity of $QH(X)$.

Additional advantage: Definition of Gromov-Witten invariants applies to more general cohomology theories.

Generalized quantum = classical theorem

$$\begin{array}{ccc} Z_d & \xrightarrow{p} & X \\ \downarrow q & & \\ Y_d & & \end{array}$$

$X = \mathrm{Gr}(m, n) = \{V\}$

$a = \max(m - d, 0)$, $b = \min(m + d, n)$

$Y_d = \mathrm{Fl}(a, b; n) = \{(A, B)\}$

$Z_d = \mathrm{Fl}(a, m, b; n) = \{(A, V, B)\}$

Thm: (B & Mihalcea)

Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$. Then

$$\int_{\overline{\mathcal{M}}_{0,3}(X,d)}^{T} \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) = \int_{Y_d}^{T} q_*(p^*(\alpha)) \cdot q_*(p^*(\beta)) \cdot q_*(p^*(\gamma))$$

Generalized quantum = classical theorem

$$\begin{array}{ccc} & & M_d \\ & & \Big\downarrow \mathrm{ev}_i \\ Z_d & \xrightarrow{\;p\;} & X \\ \Big\downarrow q & & \\ Y_d & & \end{array}$$

$M_d = \overline{\mathcal{M}}_{0,3}(X, d)$

$X = \mathrm{Gr}(m, n) = \{V\}$

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Thm: (B & Mihalcea)

Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$. Then

$$\int^T_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) = \int^T_{Y_d} q_*(p^*(\alpha)) \cdot q_*(p^*(\beta)) \cdot q_*(p^*(\gamma))$$

Generalized quantum = classical theorem

$$\begin{array}{ccccc}
\mathrm{B}\ell_d & \xrightarrow{\quad\pi\quad} & & & M_d \\
\Big\downarrow{\scriptstyle\phi} & & & & \Big\downarrow{\scriptstyle\mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
& & \Big\downarrow{\scriptstyle q} & & \\
& & Y_d & &
\end{array}$$

$$M_d = \overline{\mathcal{M}}_{0,3}(X, d)$$

$$X = \mathrm{Gr}(m, n) = \{V\}$$

$$a = \max(m - d, 0)\ , \quad b = \min(m + d, n)$$

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$$Z_d = \mathrm{Fl}(a, m, b; n) = \{(A, V, B)\}$$

$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \ : \\ A \subset V_i \subset B \end{array} \right\}$$

$$\pi(f, A, B) = f$$

$$\phi(f, A, B) = (\mathrm{ev}_1(f), \mathrm{ev}_2(f), \mathrm{ev}_3(f), A, B)$$

$$e_i(V_1, V_2, V_3, A, B) = (A, V_i, B)$$

Generalized quantum = classical theorem

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Facts:

(1) π is birational. (A general curve has kernel and span of expected dimensions.)

(2) $Z_d^{(3)} = Z_d \times_{Y_d} Z_d \times_{Y_d} Z_d$

Generalized quantum = classical theorem

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Fibers of ϕ :

Let $(A, B) \in Y_d$.

$X' = \mathrm{Gr}(m - a, B/A) = \{V \in X \mid A \subset V \subset B\}$

$(q e_i \phi)^{-1}(A, B) \;=\; \overline{\mathcal{M}}_{0,3}(X', d)$

Generalized quantum = classical theorem

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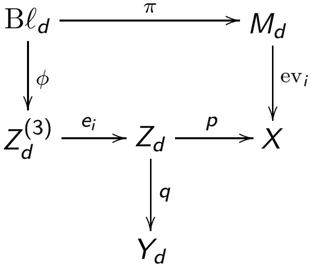
$(qe_i\phi)^{-1}(A, B) \ = \ \overline{\mathcal{M}}_{0,3}(X', d)$

Let $z = (V_1, V_2, V_3, A, B) \in Z_d^{(3)}$

$\phi^{-1}(z) \ = \ \mathrm{ev}_1^{-1}(V_1) \cap \mathrm{ev}_2^{-1}(V_2) \cap \mathrm{ev}_3^{-1}(V_3) \ \subset \overline{\mathcal{M}}_{0,3}(X', d)$

This is a Gromov-Witten variety of curves meeting 3 points in X'.

Generalized quantum = classical theorem



$$\mathrm{B}\ell_d = \left\{ \begin{matrix} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{matrix} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{matrix} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \ : \\ A \subset V_i \subset B \end{matrix} \right\}$$

Thm: (B & Mihalcea)

For all points z in a dense open subset of $Z_d^{(3)}$, the Gromov-Witten variety

$$\phi^{-1}(z) \;=\; \mathrm{ev}_1^{-1}(V_1) \cap \mathrm{ev}_2^{-1}(V_2) \cap \mathrm{ev}_3^{-1}(V_3) \quad \subset \overline{\mathcal{M}}_{0,3}(X', d)$$

is rational.

Question: Is this true for other spaces $X = G/P$?

(See papers by Chaput–Perrin and de Jong–He–Starr.)

Generalized quantum = classical theorem

$$\begin{array}{ccccc} \mathrm{B}\ell_d & & \xrightarrow{\pi} & & M_d \\ \downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \mathrm{ev}_i} \\ Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\ & & \downarrow{\scriptstyle q} & & \\ & & Y_d & & \end{array}$$

$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

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Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$.

$$\int_{M_d}^T \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) \; = \; \int_{\mathrm{B}\ell_d}^T (\mathrm{ev}_1\,\pi)^*(\alpha) \cdot (\mathrm{ev}_2\,\pi)^*(\beta) \cdot (\mathrm{ev}_3\,\pi)^*(\gamma)$$

Generalized quantum = classical theorem

$$\begin{array}{ccccc}
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Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$.

$$\begin{aligned}
\int_{M_d}^T \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) &= \int_{\mathrm{B}\ell_d}^T (\mathrm{ev}_1 \pi)^*(\alpha) \cdot (\mathrm{ev}_2 \pi)^*(\beta) \cdot (\mathrm{ev}_3 \pi)^*(\gamma) \\
&= \int_{Z_d^{(3)}}^T e_1^*(p^*\alpha) \cdot e_2^*(p^*\beta) \cdot e_3^*(p^*\gamma)
\end{aligned}$$

Generalized quantum = classical theorem

$$
\begin{array}{ccccc}
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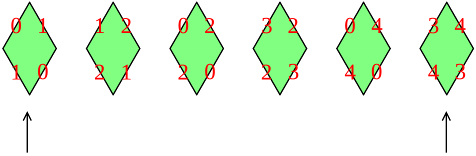
$$
\begin{aligned}
\int_{M_d}^T \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) \ &= \ \int_{\mathrm{B}\ell_d}^T (\mathrm{ev}_1\,\pi)^*(\alpha) \cdot (\mathrm{ev}_2\,\pi)^*(\beta) \cdot (\mathrm{ev}_3\,\pi)^*(\gamma) \\
&= \ \int_{Z_d^{(3)}}^T e_1^*(p^*\alpha) \cdot e_2^*(p^*\beta) \cdot e_3^*(p^*\gamma) \\
&= \ \int_{Y_d}^T q_* p^*(\alpha) \cdot q_* p^*(\beta) \cdot q_* p^*(\gamma)
\end{aligned}
$$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

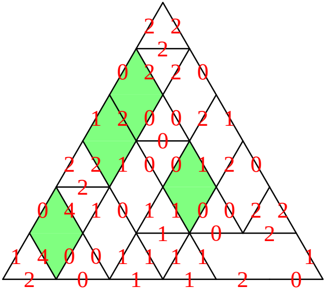
Equivariant pieces: (May NOT be rotated.)



Knutson & Tao

Surprising (3=01 , 4=12)

Equivariant puzzle:



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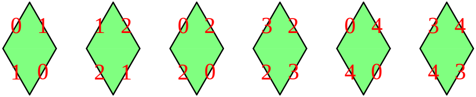
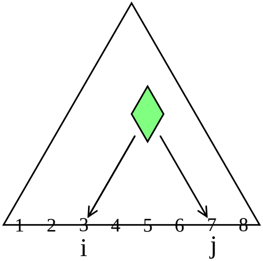


Conjecture for $H_T^*(X)$ (Buch, printed in Coskun–Vakil's 2006 survey)

$c_{u,v}^{w} = \sum_{P} \prod$ weight()

 $\in P$

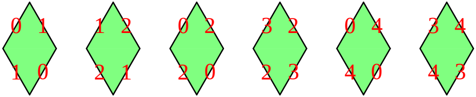
sum over equivariant puzzles P
with border labels u, v, w.



weight() $= y_j - y_i$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

Equivariant pieces: (May NOT be rotated.)

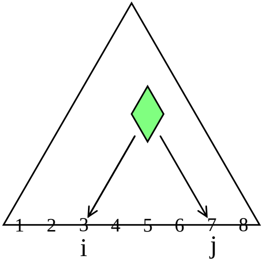


Conjecture for $H_T^*(X)$ (Buch, printed in Coskun–Vakil's 2006 survey)

$$c_{u,v}^{w} \;=\; \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

sum over equivariant puzzles P
with border labels u, v, w.

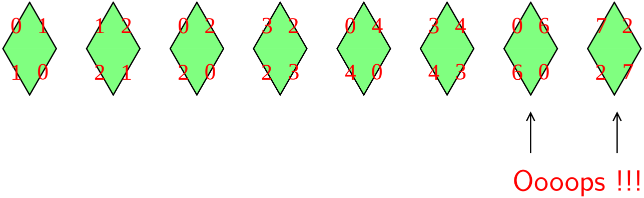
FALSE ! ! !



$\mathrm{weight}(\lozenge) = y_j - y_i$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

Equivariant pieces: (May NOT be rotated.)



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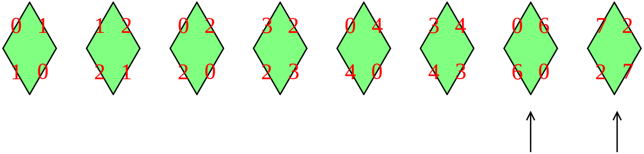

Oooops !!!

Theorem (Buch)

$$c_{u,v}^{w} \;=\; \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

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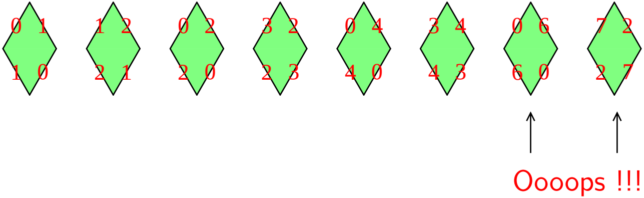

Theorem (Buch)

$$c_{u,v}^{w} \;=\; \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

Consequence: Equivariant quantum Littlewood-Richardson rule for $QH_T(\mathrm{Gr}(m, n))$.

This uses [Buch–Mihalcea 2011].

Exercise:

Let R be an associative ring with unit 1.
Let $S \subset R$ be a subset that generates R as a $\mathbb{Z}$-algebra.
Let M be a left R-module.
Let $\mu : R \times M \to M$ be any $\mathbb{Z}$-bilinear map.
Assume that for all $r \in R$, $s \in S$, and $m \in M$ we have

(1) $\mu(1, m) = m$ and

(2) $\mu(rs, m) = \mu(r, sm)$.

Then $\mu(r, m) = rm$ for all $r \in R$ and $m \in M$.

Exercise:

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Then $\mu(r, m) = rm$ for all $r \in R$ and $m \in M$.

Application:

Def: $C^w_{u,v} = \#$ puzzles with border labels u, v, w.

Def: $\mu : H^*(Y) \times H^*(Y) \to H^*(Y)$ by

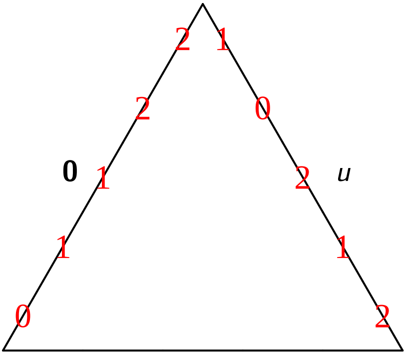
$$\mu([Y_u],[Y_v]) = \sum_w C^w_{u,v} [Y_w]$$

Enough to show: $\mu([Y_u],[Y_v]) = [Y_u] \cdot [Y_v]$

Multiplication by 1

$\mathbf{0} = 0^a\, 1^{b-a}\, 2^{n-b} = \underbrace{0000}_{a}\underbrace{11111}_{b-a}\underbrace{2222}_{n-b}$; $[Y_{\mathbf{0}}] = 1 \in H^*(Y)$

Claim: $\mu(1, [Y_u]) = [Y_u]$



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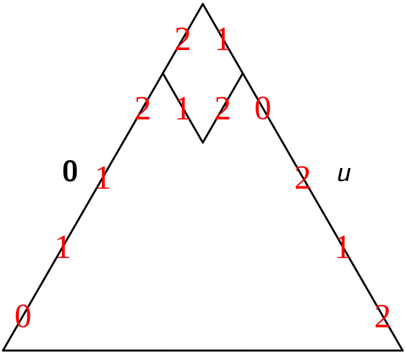

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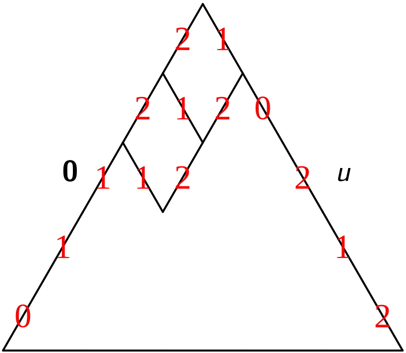

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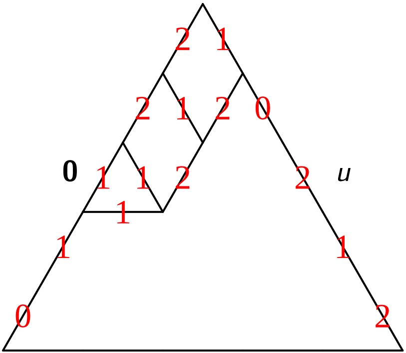

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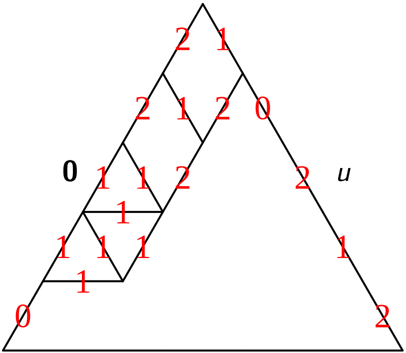

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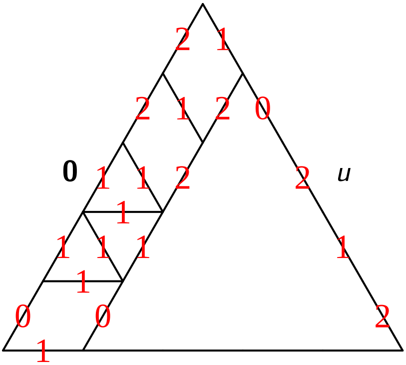

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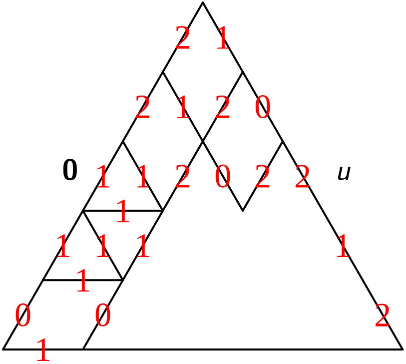

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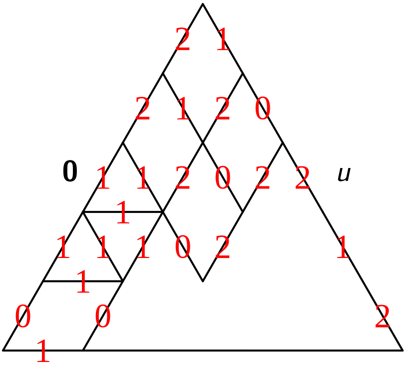

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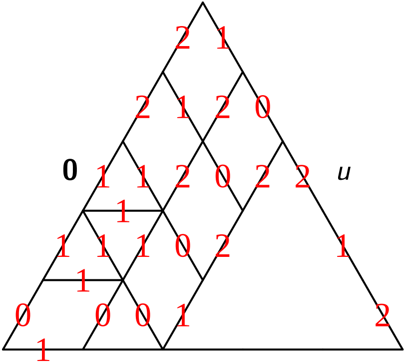

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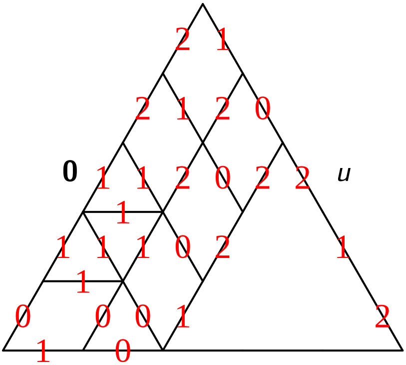
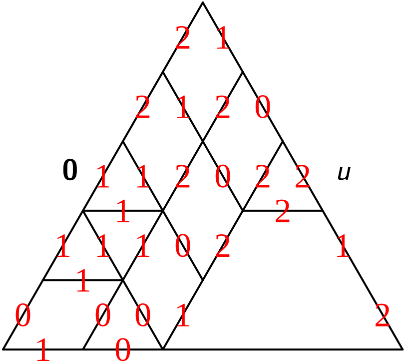


Multiplication by 1

$\mathbf{0} = 0^a\,1^{b-a}\,2^{n-b} = \underbrace{0000}_{a}\underbrace{11111}_{b-a}\underbrace{2222}_{n-b}$; $[Y_{\mathbf{0}}] = 1 \in H^*(Y)$

Claim: $\mu(1, [Y_u]) = [Y_u]$



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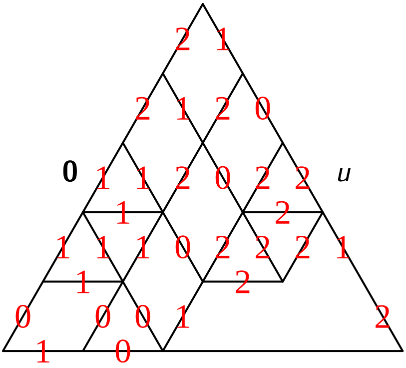

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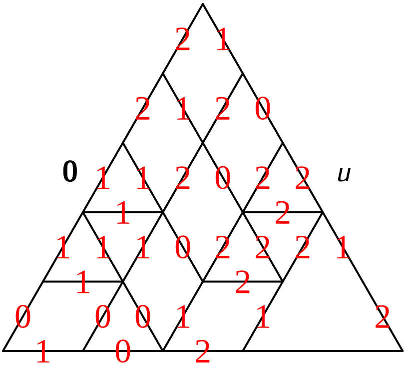

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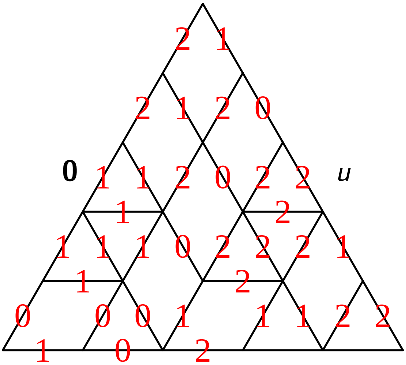

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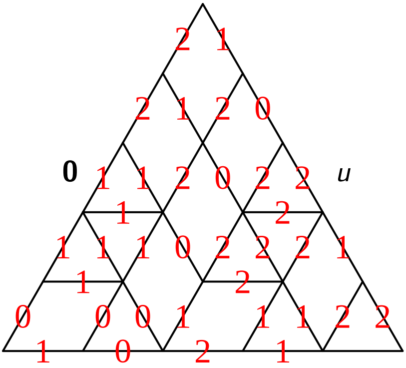

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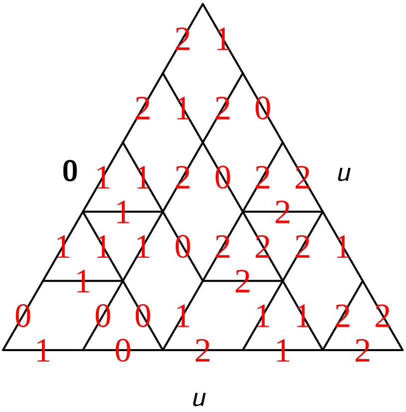

Pieri rule

Let u and u' be 012-strings.

Def: Write $u \xrightarrow{1} u'$ if u' is obtained from u by a substitution

$$02 \mapsto 20 \qquad \text{or} \qquad 100\ldots02 \mapsto 200\ldots01$$

Def: $u \xrightarrow{1} u'$ has **index** (i, j) if $i < j$ and $u_i \neq u'_i$ and $u_j \neq u'_j$.

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Def: $u \xrightarrow{1} u'$ has **index** (i, j) if $i < j$ and $u_i \neq u'_i$ and $u_j \neq u'_j$.

Let $r \in \mathbb{N}$.

Def: Write $u \xrightarrow{r} u'$ if $\exists\ u = u^0 \xrightarrow{1} u^1 \xrightarrow{1} \cdots \xrightarrow{1} u^r = u'$ such that if $u^{t-1} \xrightarrow{1} u^t$ has index (i_t, j_t) then $j_{t-1} \leq i_t$ for each t.

Example: $12021022 \xrightarrow{3} 22011202$ because:

$$12021022 \xrightarrow{1}$$
$$21021022 \xrightarrow{1}$$
$$22011022 \xrightarrow{1}$$
$$22011202$$

Pieri rule: $Y = \mathrm{Fl}(a, b; n)$

Given $r \in [0, n-b]$, identify $r = \underbrace{00000}_{a}\underbrace{1111}_{b-a-1}\underbrace{222}_{r}\mathbf{1}\underbrace{222}_{n-b-r}$

Given $p \in [0, a]$, define $\widetilde{p} = \underbrace{000}_{a-p}\mathbf{1}\underbrace{000}_{p}\underbrace{1111}_{b-a-1}\underbrace{22222}_{n-b}$

Special Schubert classes:

$[Y_r] = c_r(\mathcal{B}/\mathbb{C}^n_Y)$ and $[Y_{\widetilde{p}}] = c_p(\mathcal{A}^\vee)$ where $\mathcal{A} \subset \mathcal{B} \subset \mathbb{C}^n_Y = \mathbb{C}^n \times Y$ tautological flag on Y.

$H^*(Y)$ is generated by $S = \{[Y_1], [Y_2], \ldots, [Y_{n-b}],\ [Y_{\widetilde{1}}], [Y_{\widetilde{2}}], \ldots, [Y_{\widetilde{a}}]\}$

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Theorem: (Lascoux–Schützenberger 1982, Sottile 1996):

$$[Y_r] \cdot [Y_u] \;=\; \sum_{u \xrightarrow{r} u'} [Y_{u'}]$$

Similar formula for $[Y_{\widetilde{p}}] \cdot [Y_u]$

Must show: For each $[Y_r] \in S$ and 012-strings u and v, we have

$$\mu([Y_u] \cdot [Y_r]\ ,\ [Y_v])\ =\ \mu([Y_u]\ ,\ [Y_r] \cdot [Y_v])$$

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$$\Leftrightarrow \quad \sum_{u \xrightarrow{r} u'} C^w_{u',v} \ = \ \sum_{v \xrightarrow{r} v'} C^w_{u,v'}$$

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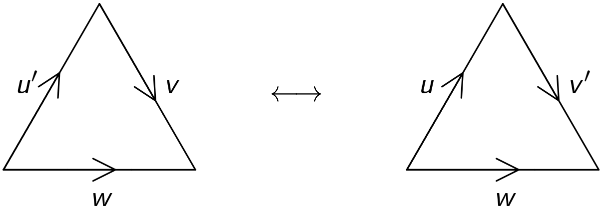
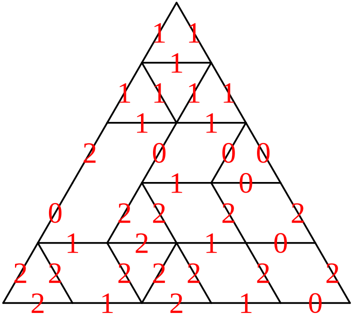
TODO: Given 012-strings u, v, w, enough to construct bijection between puzzles with border u', v, w such that $u \xrightarrow{r} u'$, and puzzles with border u, v', w such that $v \xrightarrow{r} v'$.



Easiest case: Assume $r = 1$.

$u =$ 20121 , $v =$ 11022.



$\longleftrightarrow$?

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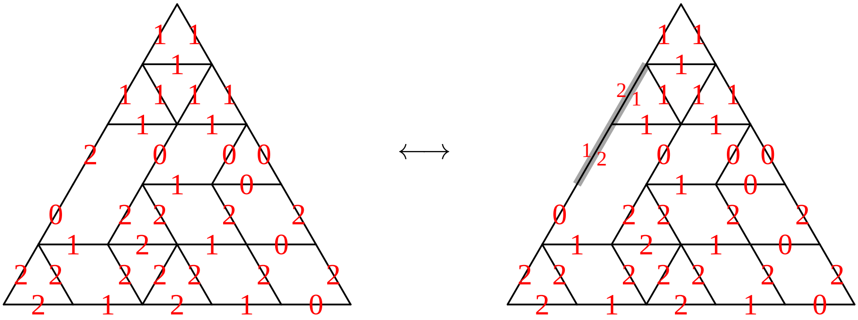

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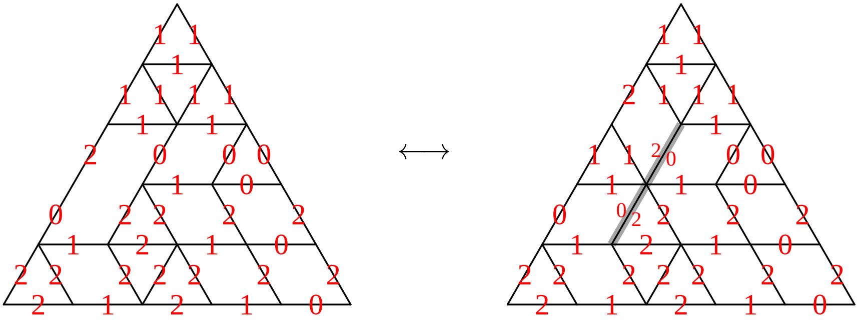

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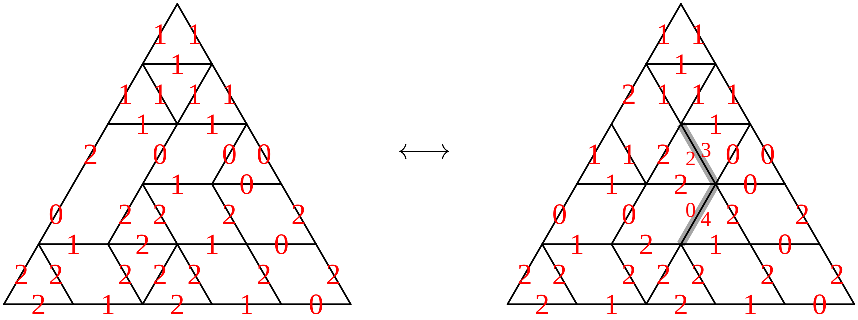

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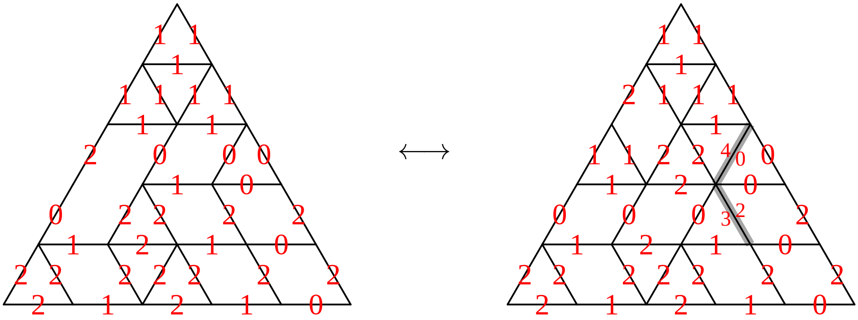

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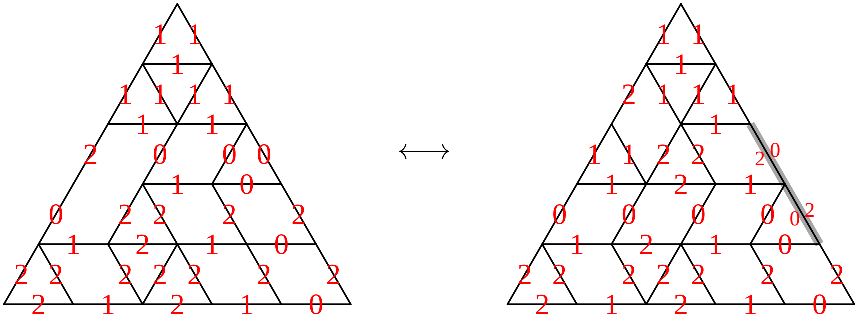

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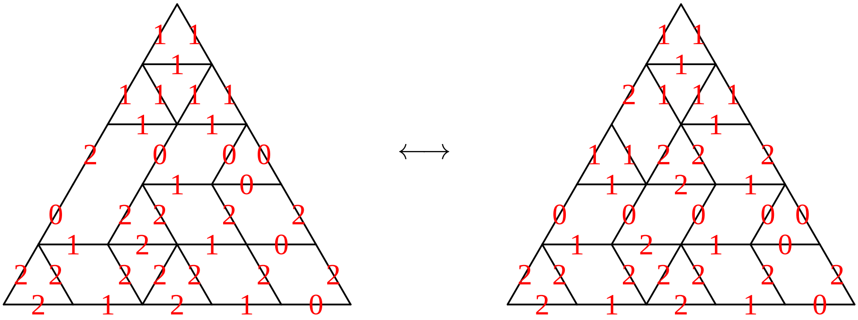
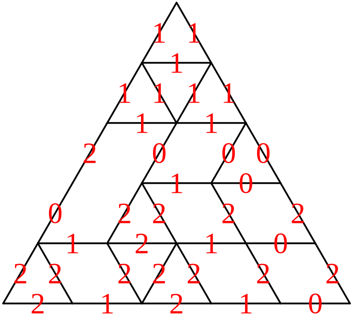


OK !

Next case: Assume $r = 2$.

$u = 10221$, $v = 11022$.



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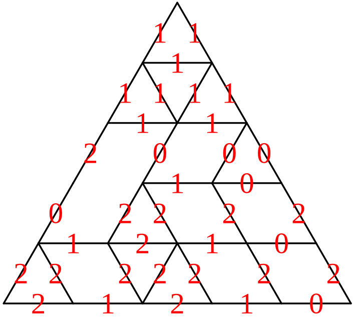

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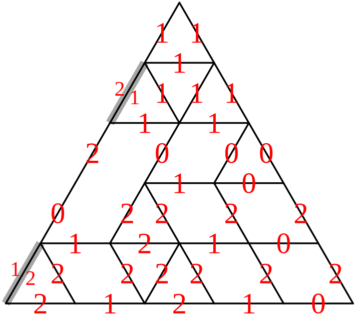

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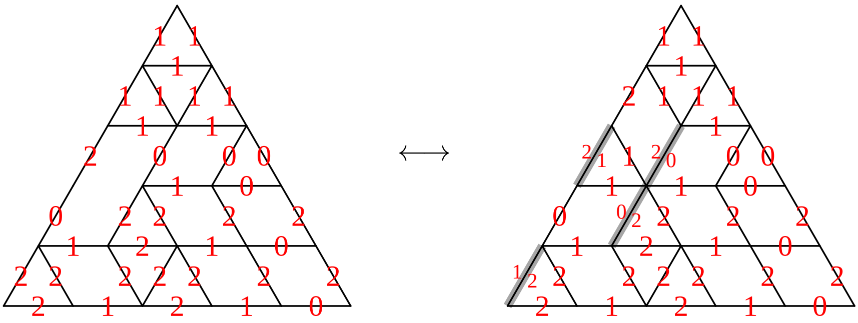

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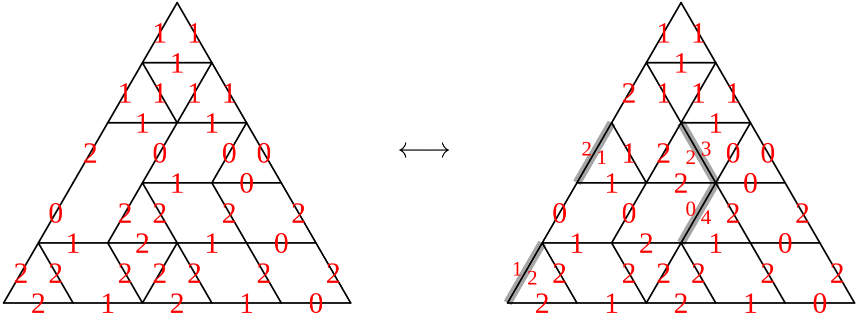

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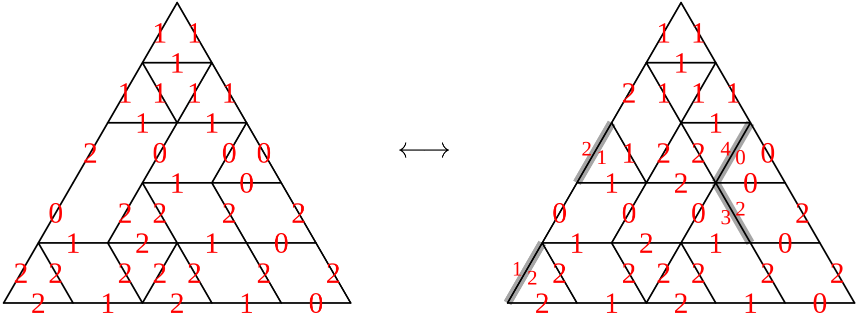

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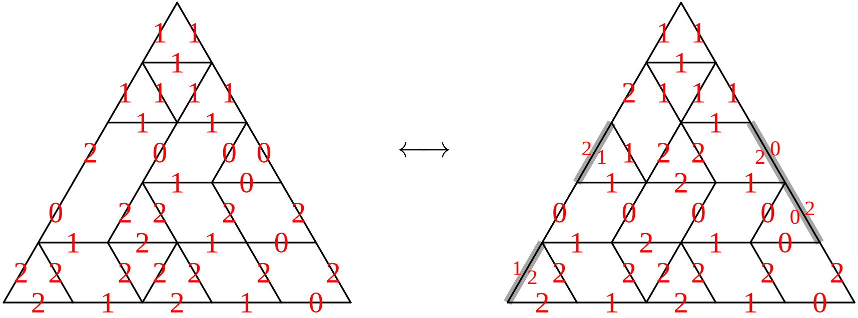

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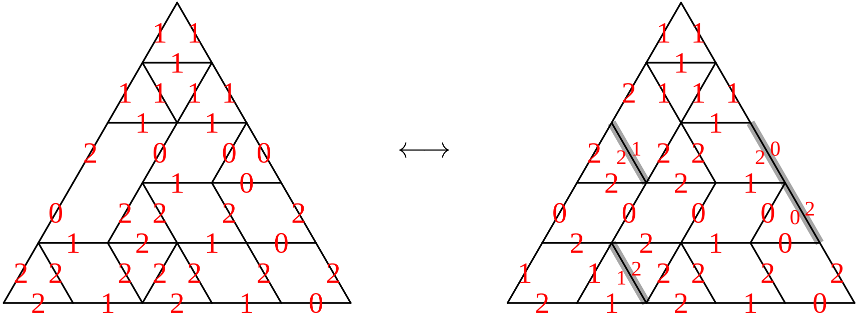

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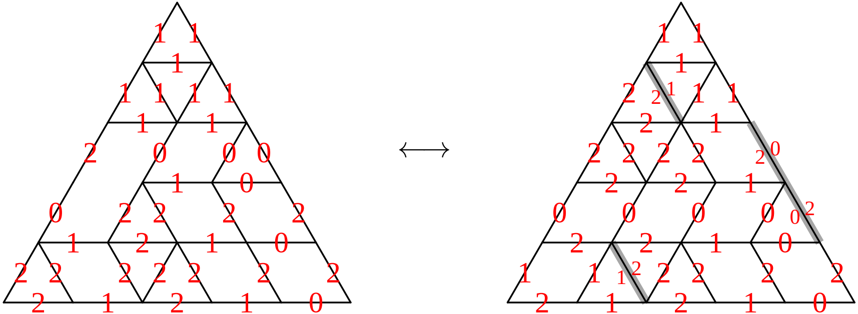

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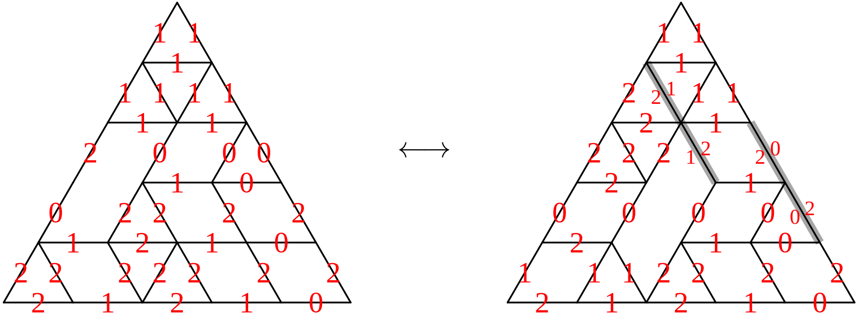

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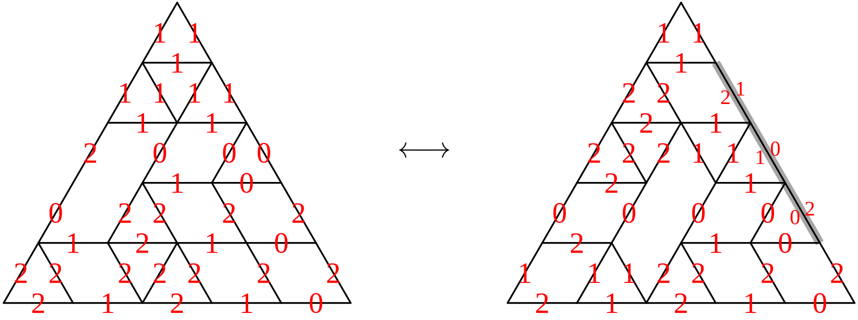

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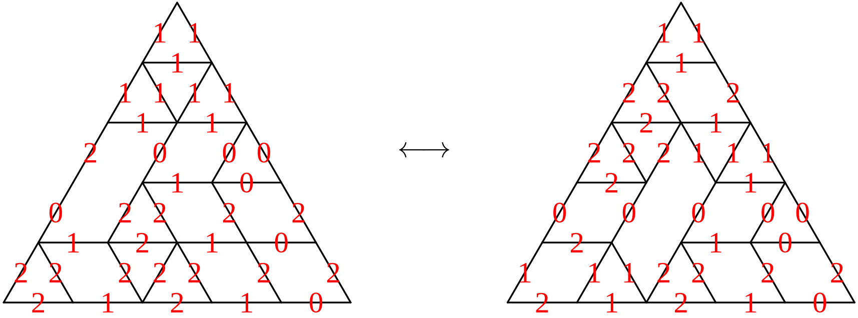

Problem: We have $v' =$ 12102, but $v \overset{2}{\not\to} v'$.

Towards a solution: Assume $r = 2$.

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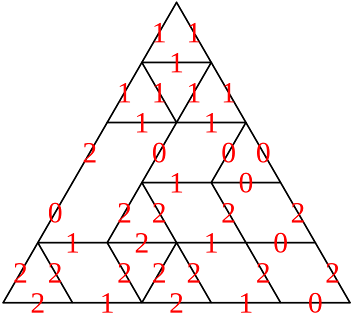

$\longleftrightarrow$

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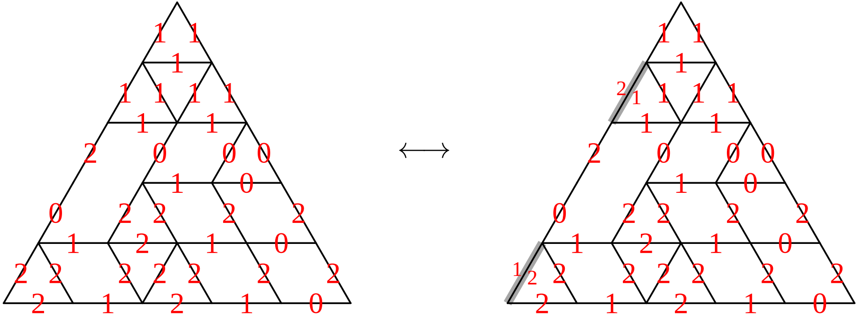

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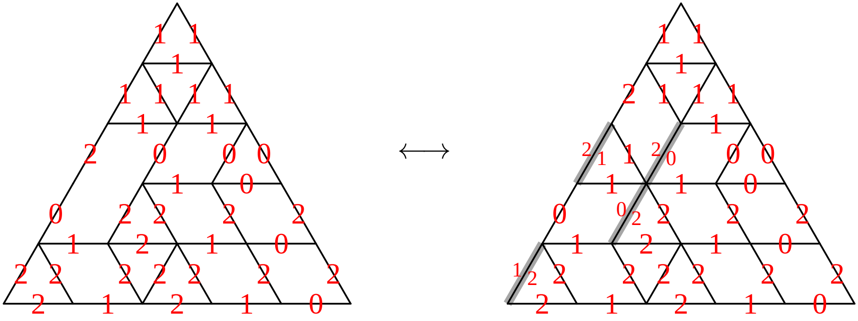

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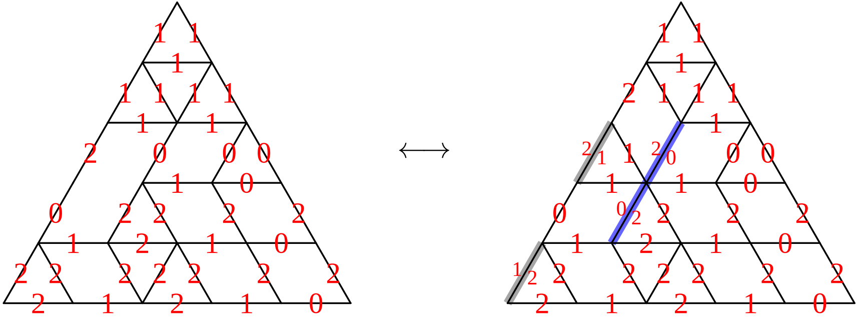

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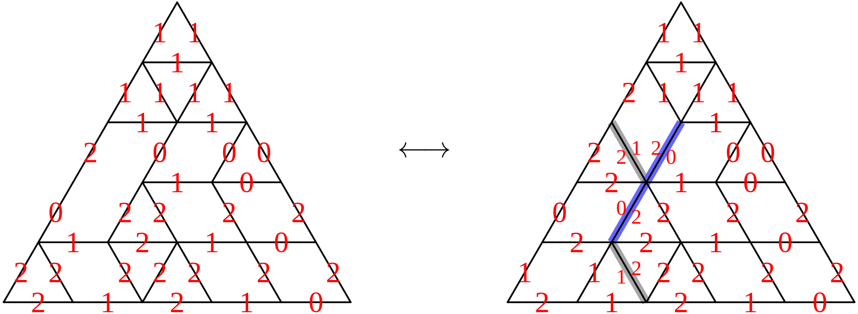

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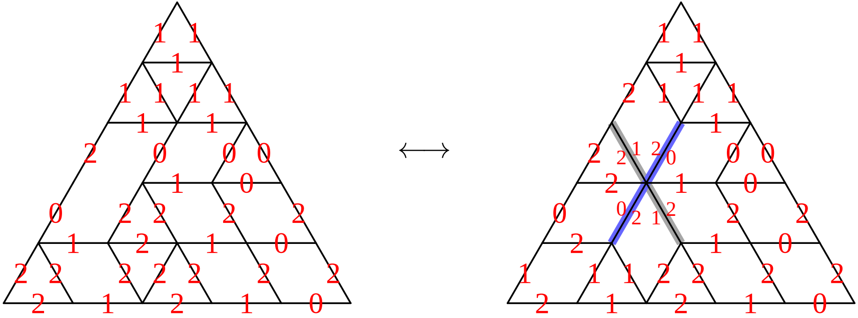

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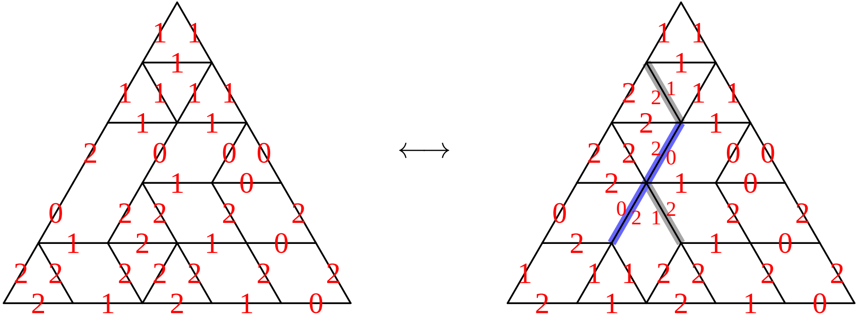

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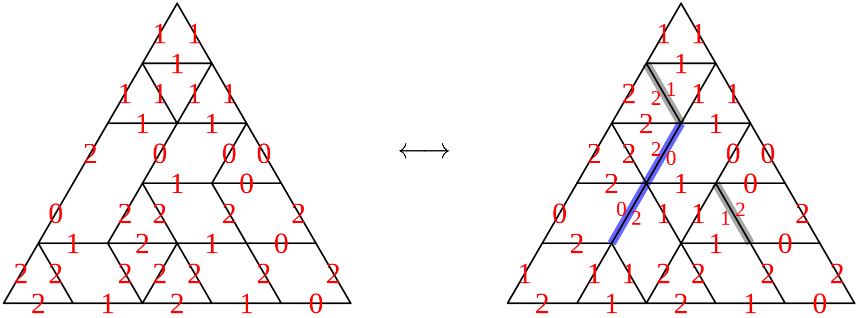

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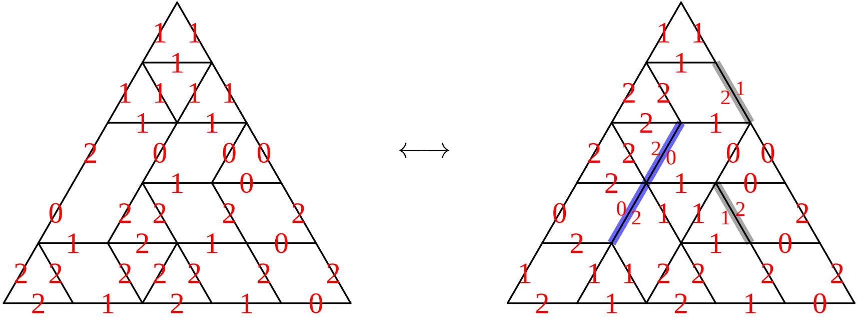

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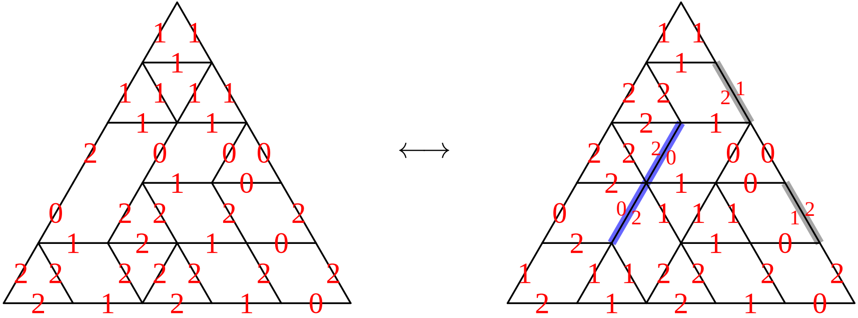

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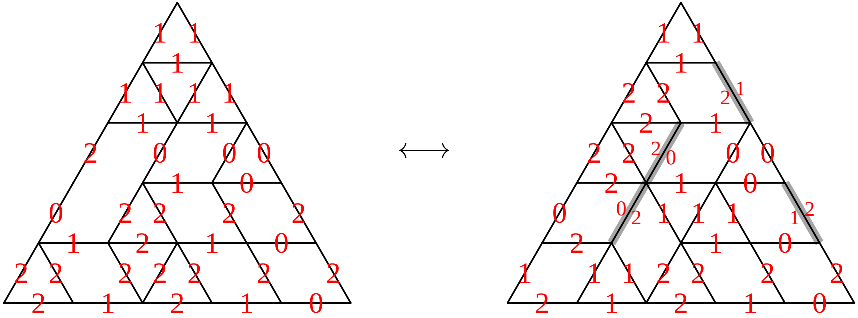

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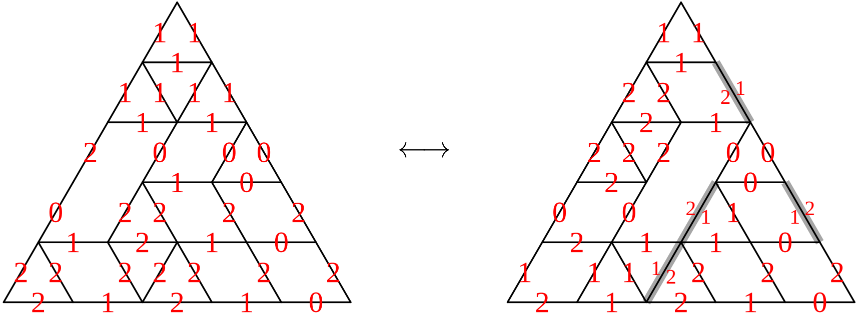

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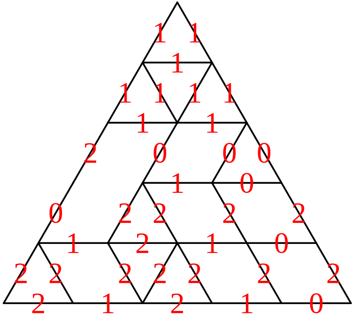
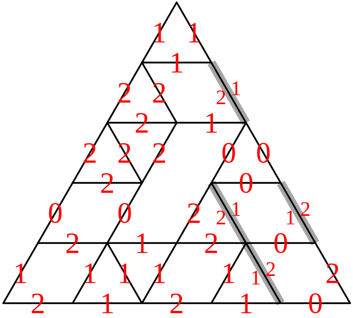


$\longleftrightarrow$



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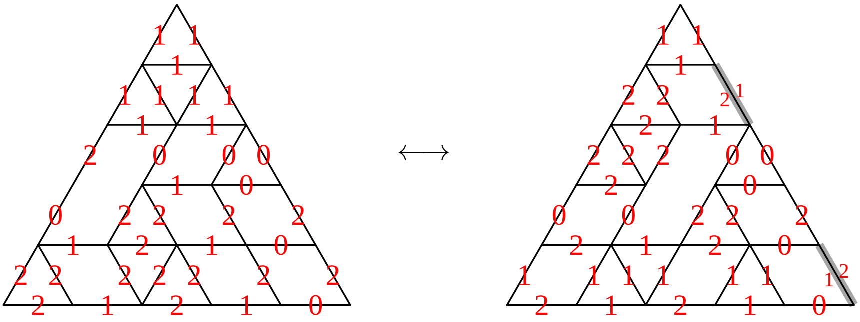

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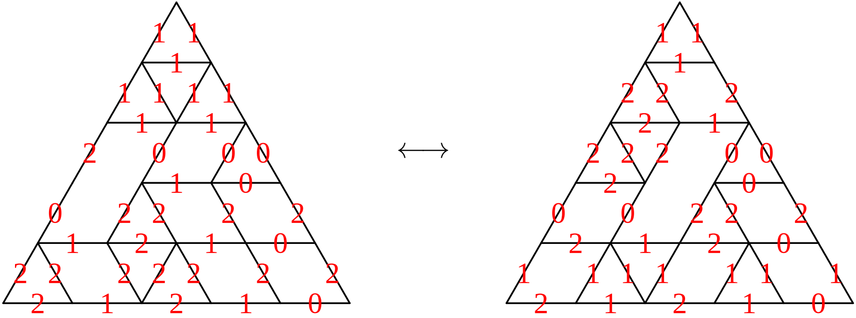
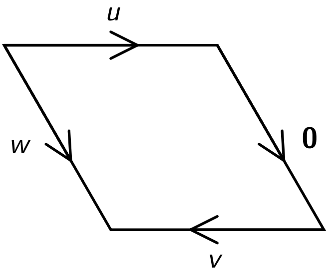


This time we have $v' = 12021$ and $v \xrightarrow{2} v'$. **OK !!**

Lemma: $C^{w}_{u,v} = \#$ rhombus shaped puzzles with border u, $\mathbf{0}$, v, w:



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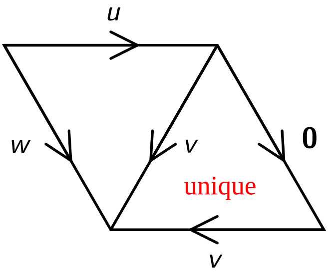
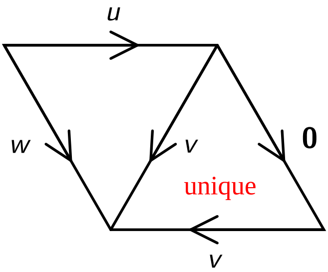
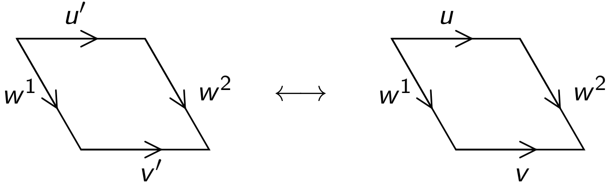


Lemma: $C_{u,v}^{w} = \#$ rhombus shaped puzzles with border u, $\mathbf{0}$, v, w:



TODO: Given 012-strings u, v', w^1, w^2, and $r \in \mathbb{N}$, construct bijection between puzzles with border (w^1, u', v', w^2) such that $u \xrightarrow{r} u'$, and puzzles with border (w^1, u, v, w^2) such that $v \xrightarrow{r} v'$.



Generalized Pieri relation

Def: A **label string** is any finite sequence of integers from $[0, 7] = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

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Def: Write $u \xrightarrow{\mathcal{R}} u'$ if u' is obtained from u by a substitution

$(a_1, s_1, \ldots, s_k, a_2) \mapsto (b_1, s_1, \ldots, s_k, b_2)$, where $s_j \in S$.

We say $u \xrightarrow{\mathcal{R}} u'$ has index (i,j) if $i < j$ and $u_i \neq u'_i$ and $u_j \neq u'_j$.

Example: $\mathcal{R} = \frac{1}{2} \; 03* \; \frac{5}{7}$ Then $704\mathbf{13035}62 \xrightarrow{\mathcal{R}} 704\mathbf{23037}62$

Index: $(4,8)$

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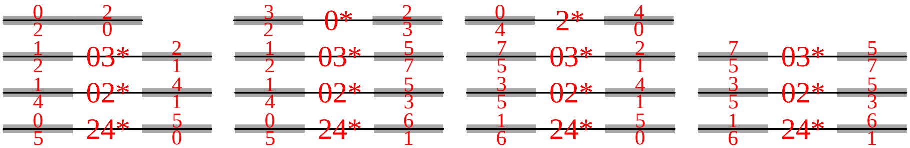
Example: $\mathcal{R} = \frac{1}{2} \; 03* \; \frac{5}{7}$ Then $704\mathbf{130356}2 \xrightarrow{\mathcal{R}} 704\mathbf{230376}2$

Index: $(4,8)$

Def: Write $u \xrightarrow{1} u'$ iff $u \xrightarrow{\mathcal{R}} u'$ for some rule $\mathcal{R}$ from the following list:

$\frac{0}{2} \; \frac{2}{0}$	$\frac{3}{2} \; 0* \; \frac{2}{3}$	$\frac{0}{4} \; 2* \; \frac{4}{0}$	
$\frac{1}{2} \; 03* \; \frac{2}{1}$	$\frac{1}{2} \; 03* \; \frac{5}{7}$	$\frac{7}{5} \; 03* \; \frac{2}{1}$	$\frac{7}{5} \; 03* \; \frac{5}{7}$
$\frac{1}{4} \; 02* \; \frac{4}{1}$	$\frac{1}{4} \; 02* \; \frac{5}{3}$	$\frac{3}{5} \; 02* \; \frac{4}{1}$	$\frac{3}{5} \; 02* \; \frac{5}{3}$
$\frac{0}{5} \; 24* \; \frac{5}{0}$	$\frac{0}{5} \; 24* \; \frac{6}{1}$	$\frac{1}{6} \; 24* \; \frac{5}{0}$	$\frac{1}{6} \; 24* \; \frac{6}{1}$

Basic rules:



Def: Write $u \xrightarrow{r} u'$ iff $\exists\ u = u^0 \xrightarrow{1} u^1 \xrightarrow{1} \cdots \xrightarrow{1} u^r = u'$, such that if $u^{t-1} \xrightarrow{1} u^t$ has index (i_t, j_t), then $j_1 < j_2 < \cdots < j_r$.

Example: 04730202245 $\xrightarrow{5}$ 40720522015 because:

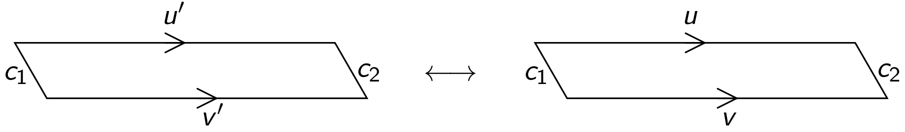
04730202245 $\xrightarrow{1}$
40730202245 $\xrightarrow{1}$
40720302245 $\xrightarrow{1}$
40720320245 $\xrightarrow{1}$
40720322045 $\xrightarrow{1}$
40720522015

Exercise:

This relation restricts to the classical Pieri relation on 012-strings.

Main Technical Result:

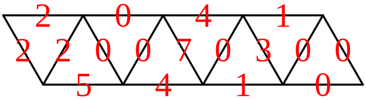
Let u and v' be label strings, let $c_1, c_2 \in \{0, 1, 2\}$, and let $r \in \mathbb{N}$.
There is an explicit bijection between
single-row puzzles with border (c_1, u', v', c_2) such that $u \xrightarrow{r} u'$, and
single-row puzzles with border (c_1, u, v, c_2) such that $v \xrightarrow{r} v'$.



Method: Propagate one gash at the time. 80 rules are required.

Example of single propagation:

$u = 0241\ ,\quad v = 5410\ ,\quad r = 1.$



$\longleftrightarrow$ **?**

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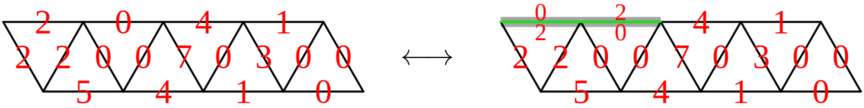

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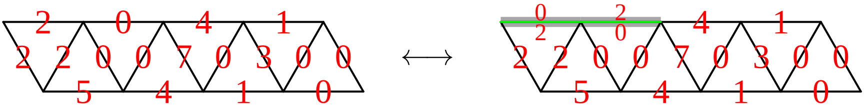
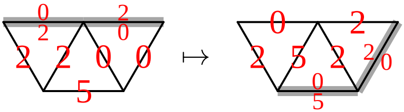


Propagation rules:



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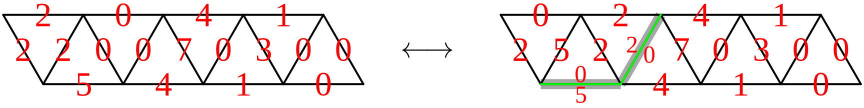

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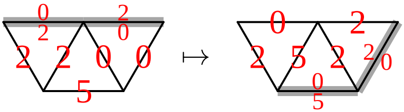

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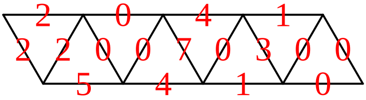
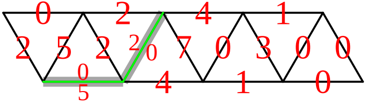


$\longleftrightarrow$



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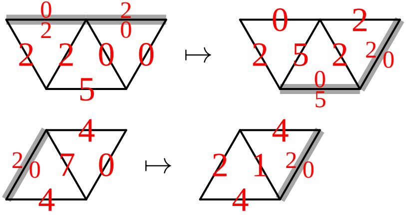

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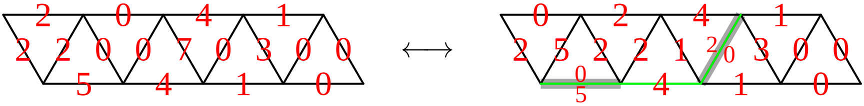

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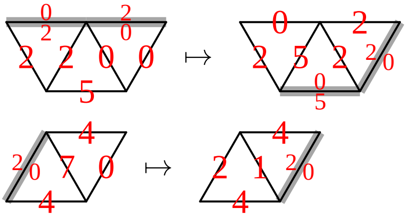

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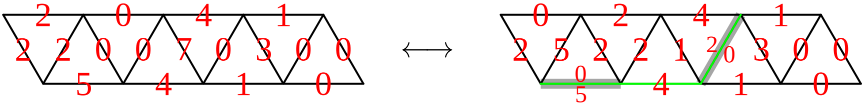

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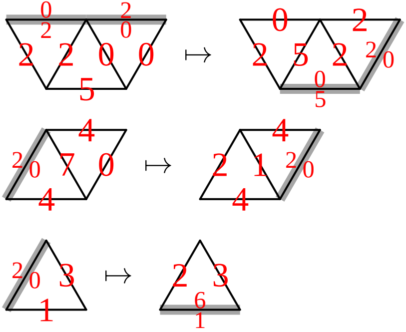

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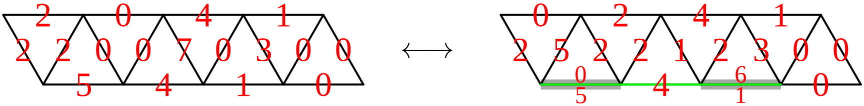

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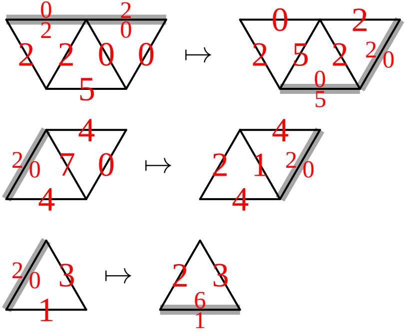

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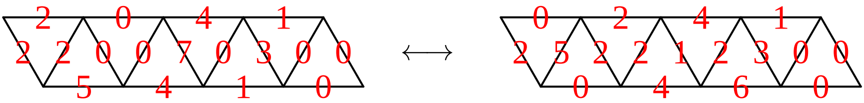
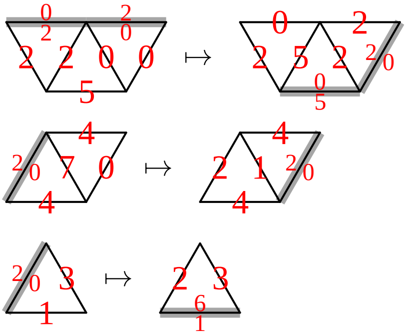


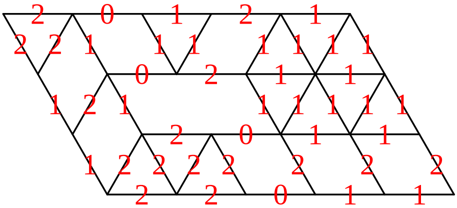
Propagation rules:

77 additional rules.



Example of resulting bijection:

$u = 10212$, $v = 22011$, $r = 2$.



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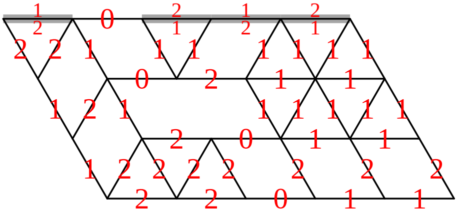

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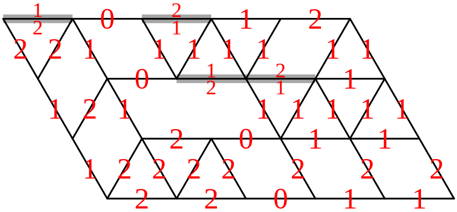

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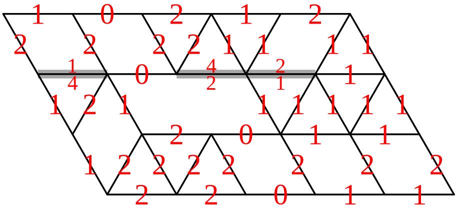

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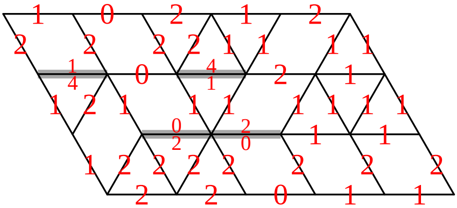

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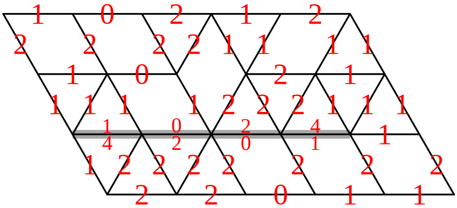

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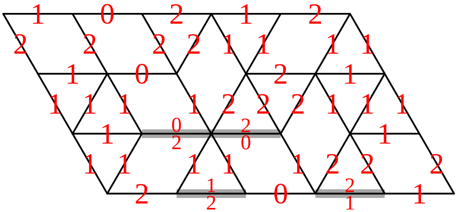

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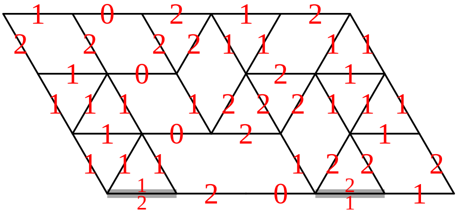

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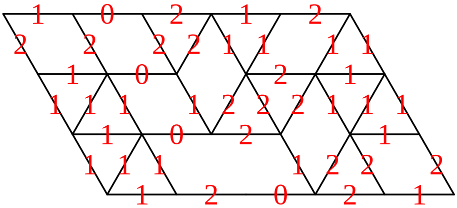

Question: What does the braid group element mean?